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DEFORMATION ANALYSIS FOR FINITE ELASTIC–PLASTIC STRAINS IN A LAGRANGEAN-TYPE DESCRIPTION

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Abstract—A general concept is presented to analyse the deformation of structures undergoing arbitrarily large elastic and arbitrarily large plastic strains. Based on the multiplicative decomposition of the deformation gradient into elastic and plastic contributions the kinematics of two superposed finite, non-coaxial deformations are investigated. Lagrangean-type elastic and plastic stretch tensors are introduced and multiplicative decompositions of the total stretch into these elastic and plastic stretches are derived. It is shown that the result is independent of any decomposition of the total rotation into an elastic and a plastic rotation.

For the second, superposed deformation the total Lagrangean logarithmic (Hencky) strain tensor with corresponding elastic and plastic logarithmic strains is defined. If in a large deformation analysis the first deformation is updated such that the second deformation is constrained to be moderately large, then the total logarithmic strain tensor of the second deformation can be additively decomposed into purely elastic and purely plastic parts. This enables an appropriate formulation of constitutive equations for isotropic hyperelastic material behavior with associated flow rule and evolution laws for combined isotropic-kinematic hardening. Work-conjugate to the elastic logarithmic strain tensor is a "back-rotated" Kirchhoff stress tensor. The rotational change of its reference configuration during the update is given explicitly.

Finally the principle of virtual work with corresponding equilibrium equations and static and geometric boundary conditions is given. The virtual work functional is transformed to deliver the consistent tangent stiffness matrix as basis for a finite element solution algorithm.

NOMENCLATURE

The following notations are used throughout the paper:

$\mathbf{F}$	deformation gradient
$\mathbf{U}$	right Cauchy–Green stretch tensor
$\mathbf{R}$	rotation tensor
$\mathbf{E}$	Green strain tensor
$\mathbf{e}$	Almansi strain tensor
$\mathbf{H}$	Lagrangean logarithmic strain tensor
$\boldsymbol{\tau}$	Kirchhoff stress tensor
$\boldsymbol{\sigma}$	Cauchy stress tensor
$(\ )^{e}$	reference to an elastic deformation
$(\ )^{p}$	reference to a plastic deformation
$(^{1}), (^{2})$	reference to a first and a second superposed deformation, respectively
$(^{\circ})$	reference to a back-rotated configuration
$(^{10}), (^{20})$	reference to a back-rotated first and second deformation, respectively, according to Section 2
$(^{1*}), (^{2*})$	reference to a transformed first and second deformation, respectively, according to Section 2
$\mathbf{AB}$	composition of the two tensors $\mathbf{A}$ and $\mathbf{B}$
$\mathbf{A}^{T}$	transpose of $\mathbf{A}$
$\mathbf{A}^{2}$	square of $\mathbf{A}$
$\mathbf{A}^{-1}$	inverse of $\mathbf{A}$.

1. INTRODUCTION

In finite elastoplasticity a straightforward generalization of the results of the linear theory with additive decomposition of elastic and plastic strains and strain rates is not possible. One concept to define plastic strains is the introduction of a local, current, stress-free intermediate configuration with an associated multiplicative decomposition of the total deformation gradient $\mathbf{F}$ into an elastic, $\mathbf{F}^{e}$, and a plastic contribution, $\mathbf{F}^{p}$, where $\mathbf{F}^{e}$ and $\mathbf{F}^{p}$ are in general incompatible point functions. This concept within finite elastoplasticity was first introduced and investigated by Lee and Liu (1967) and Lee (1969) and successfully applied by many authors [see e.g. Simo and Ortiz (1985), Simo (1988a, b) and Stumpf (1991, 1993)]. Even if this kinematical model seems to be generally accepted the identification of plastic strain is controversely discussed in the literature [e.g. Naghdi (1990)]. Two critical points should be mentiond here: (1) the existence of a stress-free state, (2) the non-uniqueness of the multiplicative decomposition of the deformation gradient and the question which invariance requirements have to be imposed on the stress-free intermediate configuration [see also Dashner (1986a) and Casey (1987)]. In the present paper it is shown that for the analysis of finite elastic–plastic strains an explicit determination of the current intermediate configuration is not needed [see also Le and Stumpf (1993)], if microstructural properties of the material behavior are not taken into account. If the stretch tensor and the Lagrangean logarithmic strain tensor, respectively, are used as appropriate strain measures, a decomposition of stretch and logarithmic strain into elastic and plastic contributions is proved to be independent of any decomposition of the rotation into an elastic and a plastic rotation, which is not unique in the frame of a Cauchy continuum.

Based on the multiplicative decomposition of the deformation gradient Lubarda and Lee (1981) derived a decomposition of the deformation rate $\mathbf{d}$ under the additional assumption that the elastic rotation $\mathbf{R}^e$ resulting from the polar decomposition of $\mathbf{F}^e$ vanishes and that $\mathbf{F}^e$ is a positive definite symmetric tensor. Under this strong restriction, which cannot be satisfied in general, they showed that in the additive decomposition the "elastic" deformation rate $\mathbf{d}^e$ is dependent on the plastic deformation and the "plastic" deformation rate $\mathbf{d}^p$ is dependent on the elastic deformation as well. This result was generalized by Stumpf and Badur (1990) without introducing any additional assumption. Using the concept of objective Lie derivatives the decomposition of the deformation rates referred to the undeformed, intermediate and deformed configuration is considered in Stumpf (1993), where it is shown that for all three pictures an additive decomposition into purely elastic and purely plastic parts is not possible. Therefore, they are not very suitable measures for formulating elastic and plastic constitutive equations and for deriving tractable solution algorithms for large strain deformation.

In many papers of finite elastoplasticity based on the kinematics of the multiplicative decomposition of the deformation gradient [e.g. Dafalias (1983)] constitutive equations are also formulated for the plastic spin $\mathbf{w}^p$ in connection with a symmetric Cauchy stress tensor. In contrast to these results Stumpf and Badur (1990) and Nemat-Nasser (1992) proved that for a Cauchy continuum the plastic spin is a function depending on $\mathbf{d}^p$ and on the elastic and total deformation, respectively. In continua with microstructure [see Le and Stumpf (1994)] separate constitutive equations have to be formulated for the plastic torsion rather than for the plastic spin.

The aim of this paper is to present kinematics and the associated constitutive model for arbitrarily large elastic and arbitrarily large plastic strains. The material behavior is assumed to be isotropically elastic and plastic with isotropic and kinematic hardening.

Dashner (1986b) investigated large strain elastic–plastic constitutive relations under thermodynamic restrictions. He derived a formula, which allows a multiplicative decomposition of the total stretch into an elastic and a plastic stretch. The Drucker-inequality and J_2-type yield functions with isotropic work hardening were considered in detail. Simo (1988a) established a framework based on a hyperelastic stress response and a reformulated flow rule in the strain space. Difficulties arise due to the fact that the strain rates employed to formulate the constitutive equations are not purely elastic and purely plastic measures, respectively. The presented incremental algorithm can realize load steps corresponding to strain increments of about 1%. By applying the logarithmic strain tensor Weber and Anand (1990) investigated the analysis of materials, which are initially isotropic and remain so during the elastic–plastic deformation. This model excludes any anisotropy induced by kinematic hardening. The logarithmic strain tensor is also used in Etorovic and Bathe (1990) to derive a constitutive model, which allows the analysis of moderately large elastic strains and plastic strains up to about 40% with isotropic and kinematic hardening. In the last three models the plastic spin is assumed to be zero.

In the analysis of this paper no assumptions or restrictions are made concerning the magnitude of elastic and plastic strains or elastic and plastic rotations and spins, respectively.

The occurrence of large plastic strains is almost obvious, especially in metal plasticity, where they often appear in combination with small elastic strains. But if the volumetric (hydrostatic) pressure is very high as in the case of high velocity impact and explosion phenomena, even metals can undergo large elastic dilatational changes. Also, considering rubber-like materials, where large elastic strains commonly occur, the long-term behavior with viscous properties can lead to large plastic strains, too. So, it seems to be worthwhile investigating finite elastic–plastic material models with simultaneous large elastic and large plastic strain deformations. The same can be said about the rotations, which are the main contributions in the non-linear deformation behavior of thin shell-like structures [see e.g. Stumpf (1986) and Pietraszkiewicz (1977, 1989)]. Therefore restrictions concerning the magnitude of elastic and plastic rotations can be made only in the frame of the structural model under consideration.

The basic assumption of our investigation is the multiplicative decomposition of the deformation gradient. With this point of departure we have to derive appropriate elastic and plastic strain and strain rate measures. First, we consider in Section 2.1 the exact kinematics of a finite, non-coaxial deformation and determine a decomposition of the total stretch into a plastic and a back-rotated elastic stretch. Special attention is paid to obtaining stretch tensors referred to fixed reference configurations allowing the use of their material time derivatives as objective rates.

In general, incremental approximation procedures are based on the superposition of an infinitesimally small deformation on a finite deformation. To avoid any shortcomings we investigate in Section 2.2 the exact kinematics of two superposed finite elastic–plastic deformations. The stretch of the second, superposed deformation is decomposed into elastic and plastic parts referred to a back-rotated first deformation. Also the total stretch is decomposed with respect to the undeformed reference configuration. Then in Section 3 the superposed deformation is constrained to comprise only moderately large elastic–plastic strains, but unrestricted rotations. Introducing the Lagrangean logarithmic (Hencky) strain tensor it is shown that within the moderate strain assumption for the superposed deformation the logarithmic strain tensor can be additively decomposed into purely elastic and purely plastic contributions, and for a fixed and known first deformation also the objective time derivative of the logarithmic strain tensor can be additively decomposed into purely elastic and purely plastic logarithmic rates. To ensure that the superposed deformation remains within the limit of moderately large strains, an update of the first deformation has to be performed, when this limit is attained. The orientational change of the reference configuration during the update is determined exactly.

In Section 4 the objective rates of the elastic and plastic logarithmic strain tensors are used to formulate appropriate constitutive equations for isotropic–hyperelastic material behavior and associated plastic flow rule for combined isotropic–kinematic hardening. The model comprises materials, which are initially elastically isotropic and remain so during the deformation process. The plastic response may be anisotropic according to continuous plastic flow. Materials of this type with no explicit account for the microstructure are considered in Casey (1987) and they are denoted materials of type A, in contrast to those of type B with a strong dependency on the microstructural concept [see also Dashner (1986a)].

It is furthermore shown that within the moderate strain assumption for the superposed deformation the stress measure work-conjugate to the Lagrangean logarithmic strain tensor is a back-rotated Kirchhoff stress tensor. Of special importance is the fact that the back-stress tensor describing kinematic hardening has to be referred to the same reference configuration as the back-rotated Kirchhoff stress tensor. Its rotational change during the update procedure has to be determined exactly.

In Section 5 the principle of virtual work is formulated and equilibrium equations and static boundary conditions are derived as Euler–Lagrange equations. Finally, the tangent stiffness matrix for a finite element procedure is presented.

2. KINEMATICS OF NON-COAXIAL ELASTIC–PLASTIC DEFORMATIONS

2.1. *One finite elastic–plastic deformation*

We consider the deformation ϕ of a body $\mathfrak{B}$ from the initial undeformed configuration $\mathring{\mathfrak{B}}$ to the deformed configuration $\mathfrak{B}$. Let $\mathbf{X}$ be the position vector of a material point in $\mathring{\mathfrak{B}}$

and $\mathbf{x} = \phi(\mathbf{X}, t)$ its position vector in the deformed configuration, where t is the time parameter. The deformation gradient $\mathbf{F}$ decribes the tangent of the deformation, $\mathbf{F} = T\phi$. Assuming that the deformation is regular then the right polar decomposition is valid,

$$\mathbf{F} = \mathbf{R}\mathbf{U}, \quad \mathbf{R}^T\mathbf{R} = \mathbf{1}, \quad \mathbf{U} = \mathbf{U}^T > 0, \tag{1}$$

where $\mathbf{R}$ is the rotation tensor and $\mathbf{U}$ the symmetric, positive definite Lagrangean stretch tensor

$$\mathbf{U}^2 = \mathbf{F}^T\mathbf{F} = \mathbf{1} + 2\mathbf{E} \tag{2}$$

and $\mathbf{E}$ the Green strain tensor.

Following Lee and Liu (1967) and Lee (1969) we use the concept of a local, current, relaxed state $\mathfrak{B}^p$ of the elastic–plastic body with stresses removed and change of temperature to a constant reference value. Then the multiplicative decomposition of the deformation gradient $\mathbf{F}$ into elastic and plastic parts $\mathbf{F}^e$ and $\mathbf{F}^p$ is valid

$$\mathbf{F} = \mathbf{F}^e\mathbf{F}^p, \tag{3}$$

where in general $\mathbf{F}^e$ and $\mathbf{F}^p$ are incompatible point functions.

Applying the polar decomposition theorem to $\mathbf{F}^e$ and $\mathbf{F}^p$ we obtain

$$\mathbf{F}^e = \mathbf{R}^e\mathbf{U}^e, \quad \mathbf{U}^{e2} = \mathbf{F}^{eT}\mathbf{F}^e, \tag{4}$$

$$\mathbf{F}^p = \mathbf{R}^p\mathbf{U}^p, \quad \mathbf{U}^{p2} = \mathbf{F}^{pT}\mathbf{F}^p \tag{5}$$

with symmetric and positive definite elastic and plastic stretches $\mathbf{U}^e$ and $\mathbf{U}^p$. A schematic sketch of the polar decompositions (1), (4) and (5) is shown in Fig. 1.

In finite elastoplasticity a crucial point is the definition of objective elastic and plastic strain rate measures. In order to avoid the need to use Lie derivatives as objective rates [see also Marsden and Hughes (1983)] we focus our attention on the definition of Lagrangean-type elastic and plastic stretch tensors referred to fixed reference configurations.

The total stretch $\mathbf{U}$ and the plastic stretch $\mathbf{U}^p$ are referred to the undeformed configuration $\mathring{\mathfrak{B}}$, while the elastic stretch $\mathbf{U}^e$ is referred to the transformed configuration $\mathfrak{B}^p$

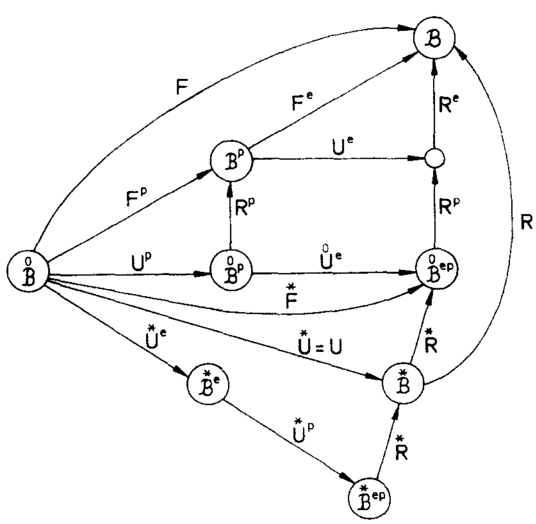

Fig. 1. Decompositions of the deformation gradient F.

(see Fig. 1). We first introduce a back-rotated elastic stretch tensor $\mathring{\mathbf{U}}^{\mathrm{e}}$ by

$$\mathring{\mathbf{U}}^{\mathrm{e}} := (\mathbf{R}^{\mathrm{p}})^{\mathrm{T}}\mathbf{U}^{\mathrm{e}}\mathbf{R}^{\mathrm{p}}, \tag{6}$$

which transforms the plastically stretched configuration $\mathring{\mathfrak{B}}^{\mathrm{p}}$ into the elastic–plastically stretched configuration $\mathfrak{B}^{\mathrm{ep}}$. With (6) and (3)–(5) the total deformation gradient $\mathbf{F}$ can be obtained in the form

$$\mathbf{F} = \mathbf{R}^{\mathrm{e}}\mathbf{U}^{\mathrm{e}}\mathbf{R}^{\mathrm{p}}\mathbf{U}^{\mathrm{p}} = \mathbf{R}^{\mathrm{e}}\mathbf{R}^{\mathrm{p}}\mathring{\mathbf{U}}^{\mathrm{e}}\mathbf{U}^{\mathrm{p}} \tag{7}$$

leading with (2) to the following decomposition of the total stretch tensor $\mathbf{U}$ into elastic and plastic parts

$$\mathbf{U}^2 = \mathbf{F}^{\mathrm{T}}\mathbf{F} = \mathbf{U}^{\mathrm{p}}\mathring{\mathbf{U}}^{\mathrm{e}2}\mathbf{U}^{\mathrm{p}}, \quad \mathring{\mathbf{U}}^{\mathrm{e}} = \sqrt{(\mathbf{U}^{\mathrm{p}})^{-1}\mathbf{U}^2(\mathbf{U}^{\mathrm{p}})^{-1}}. \tag{8}$$

The back-rotated elastic stretch tensor $\mathring{\mathbf{U}}^{\mathrm{e}}$ is in the general case of non-coaxial deformations not an appropriate measure to define objective elastic strain rates. This can be seen immediately from the fact that in eqn (7) the composition of the two symmetric stretch tensors $\mathring{\mathbf{U}}^{\mathrm{e}}$ and $\mathbf{U}^{\mathrm{p}}$ is non-symmetric and non-commutative and therefore connected with a rotation $\overset{*}{\mathbf{R}}$, which can be determined by polar decomposition

$$\overset{*}{\mathbf{F}} := \mathring{\mathbf{U}}^{\mathrm{e}}\mathbf{U}^{\mathrm{p}} = \overset{*}{\mathbf{R}}\overset{*}{\mathbf{U}} \tag{9}$$

with the total stretch tensor

$$\overset{*}{\mathbf{U}}{}^2 = \overset{*}{\mathbf{F}}{}^{\mathrm{T}}\overset{*}{\mathbf{F}} = \mathbf{U}^{\mathrm{p}}\mathring{\mathbf{U}}^{\mathrm{e}2}\mathbf{U}^{\mathrm{p}} = \mathbf{U}^2 \tag{10}$$

and the rotation tensor

$$\overset{*}{\mathbf{R}} = \mathring{\mathbf{U}}^{\mathrm{e}}\mathbf{U}^{\mathrm{p}}\sqrt{(\mathbf{U}^{\mathrm{p}})^{-1}(\mathring{\mathbf{U}}^{\mathrm{e}})^{-2}(\mathbf{U}^{\mathrm{p}})^{-1}}. \tag{11}$$

Equations (7), (9) and (10) lead to the deformation gradient $\mathbf{F}$ and a decomposition of the total rotation tensor $\mathbf{R}$ in the form

$$\mathbf{F} = \mathbf{R}\mathbf{U}, \quad \mathbf{R} = \mathbf{R}^{\mathrm{e}}\mathbf{R}^{\mathrm{p}}\overset{*}{\mathbf{R}}. \tag{12}$$

With the rotation $\overset{*}{\mathbf{R}}$ we are able to define an elastic stretch tensor $\overset{*}{\mathbf{U}}{}^{\mathrm{e}}$, referred to the undeformed configuration $\mathring{\mathfrak{B}}$, by pulling back $\mathring{\mathbf{U}}^{\mathrm{e}}$ with $\overset{*}{\mathbf{R}}$. Correspondingly we introduce a transformed plastic stretch tensor $\overset{*}{\mathbf{U}}{}^{\mathrm{p}}$,

$$\overset{*}{\mathbf{U}}{}^{\mathrm{p}} := \overset{*}{\mathbf{R}}{}^{\mathrm{T}}\mathbf{U}^{\mathrm{p}}\overset{*}{\mathbf{R}}, \quad \overset{*}{\mathbf{U}}{}^{\mathrm{e}} := \overset{*}{\mathbf{R}}{}^{\mathrm{T}}\mathring{\mathbf{U}}^{\mathrm{e}}\overset{*}{\mathbf{R}}, \tag{13}$$

where $\overset{*}{\mathbf{U}}{}^{\mathrm{p}}$ is referred to the stretched and rotated configuration $\overset{*}{\mathfrak{B}}{}^{\mathrm{ep}}$.

Using (9) and (13) we can transform (7) as follows:

$$\mathbf{F} = \mathbf{R}^{\mathrm{e}}\mathbf{R}^{\mathrm{p}}\overset{*}{\mathbf{R}}\mathbf{U}\overset{*}{\mathbf{R}}{}^{\mathrm{T}}\overset{*}{\mathbf{R}} = \mathbf{R}^{\mathrm{e}}\mathbf{R}^{\mathrm{p}}\overset{*}{\mathbf{R}}\mathbf{U}^{\mathrm{p}}\mathring{\mathbf{U}}^{\mathrm{e}}\overset{*}{\mathbf{R}} = \mathbf{R}^{\mathrm{e}}\mathbf{R}^{\mathrm{p}}\overset{*}{\mathbf{R}}{}^2\overset{*}{\mathbf{U}}{}^{\mathrm{p}}\overset{*}{\mathbf{U}}{}^{\mathrm{e}}. \tag{14}$$

From (14) we derive the decomposition of the total stretch tensor $\mathbf{U}$ into the elastic and plastic parts $\overset{*}{\mathbf{U}}{}^{\mathrm{e}}$, $\overset{*}{\mathbf{U}}{}^{\mathrm{p}}$:

$$\mathbf{U}^2 = \overset{*}{\mathbf{U}}{}^{\mathrm{e}}\overset{*}{\mathbf{U}}{}^{\mathrm{p}2}\overset{*}{\mathbf{U}}{}^{\mathrm{e}}. \tag{15}$$

Also from (14) we can see that $\overset{*}{\mathbf{U}}{}^{e}$ is referred to the undeformed configuration $\overset{\circ}{\mathfrak{B}}$ as mentioned above. Equations $(8)_1$ and (15) show that the two decompositions of the total stretch $\mathbf{U}$ are independent of any assumption about elastic, $\mathbf{R}^e$, and plastic rotation $\mathbf{R}^p$.

2.2. *Two superposed finite elastic–plastic deformations*

Efficient numerical approximation procedures are based on the superposition of two deformations, where in general the second deformation is assumed to be infinitesimally small and an update procedure is performed for the first deformation after each load step. To analyse elastic–plastic deformations undergoing arbitrarily large strains and to avoid serious shortcomings in the numerical approximation procedure we investigate in this section the exact kinematics of two superposed non-coaxial finite elastic–plastic deformations.

Let us consider a first finite elastic–plastic deformation $\overset{1}{\phi}$ with the deformation gradient $\overset{1}{\mathbf{F}}: T\overset{\circ}{\mathfrak{B}} \rightarrow T\overset{1}{\mathfrak{B}}$ and a second superposed elastic–plastic deformation $\overset{2}{\phi}$ with the deformation gradient $\overset{2}{\mathbf{F}}: T\overset{1}{\mathfrak{B}} \rightarrow T\overset{2}{\mathfrak{B}}$ such that we have the total deformation $\phi = \overset{2}{\phi} \circ \overset{1}{\phi}$ with the total deformation gradient:

$$\mathbf{F} = \overset{2}{\mathbf{F}}\overset{1}{\mathbf{F}}: \quad T\overset{\circ}{\mathfrak{B}} \rightarrow T\overset{2}{\mathfrak{B}}. \tag{16}$$

Quantities related to the first deformation $\overset{1}{\phi}$ are indicated by $(\overset{1}{\cdot})$ and quantities related to the second superposed deformation are indicated by $(\overset{2}{\cdot})$.

Let us consider in this section the kinematics of the superposed deformation with respect to a configuration $\overset{10}{\mathfrak{B}}{}^{ep}$, obtained from $\overset{1}{\mathfrak{B}}$ by pull-back with $\overset{1}{\mathbf{R}}{}^{e}\overset{1}{\mathbf{R}}{}^{p}$, where $\overset{10}{\mathfrak{B}}{}^{ep}$ corresponds to $\overset{\circ}{\mathfrak{B}}{}^{ep}$ of the previous section (see Fig. 2). Applying elastic–plastic and polar decomposition to the deformation gradient $\overset{2}{\mathbf{F}}$, we obtain analogously to (7)

$$\overset{2}{\mathbf{F}} = \overset{2}{\mathbf{R}}{}^{e}\overset{2}{\mathbf{U}}{}^{e}\overset{2}{\mathbf{R}}{}^{p}\overset{2}{\mathbf{U}}{}^{p}: \quad T\overset{1}{\mathfrak{B}} \rightarrow T\overset{2}{\mathfrak{B}}. \tag{17}$$

Referred to $\overset{10}{\mathfrak{B}}{}^{ep}$ we define the deformation gradient $\overset{20}{\mathbf{F}}$ by pulling back one leg of the two-point tensor $\overset{2}{\mathbf{F}}$ with $\overset{1}{\mathbf{R}}{}^{e}\overset{1}{\mathbf{R}}{}^{p}$:

$$\overset{20}{\mathbf{F}} := \overset{2}{\mathbf{F}}\overset{1}{\mathbf{R}}{}^{e}\overset{1}{\mathbf{R}}{}^{p}: \quad T\overset{10}{\mathfrak{B}}{}^{ep} \rightarrow T\overset{2}{\mathfrak{B}} \tag{18}$$

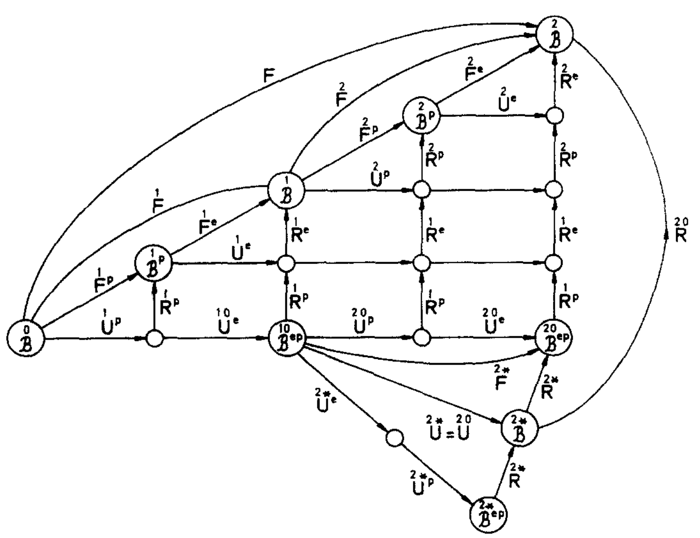

Fig. 2. Decompositions of the total deformation gradient $\mathbf{F}$ for two superposed deformations.

yielding the stretch tensor of the second superposed deformation:

$$\overset{2}{\mathbf{U}}{}^2 = \overset{2}{\mathbf{F}}{}^{\mathrm{T}}\overset{2}{\mathbf{F}}, \quad \overset{20}{\mathbf{U}}{}^2 = \overset{20}{\mathbf{F}}{}^{\mathrm{T}}\overset{20}{\mathbf{F}} = (\overset{1}{\mathbf{R}}{}^{\mathrm{p}})^{\mathrm{T}}(\overset{1}{\mathbf{R}}{}^{\mathrm{e}})^{\mathrm{T}}\overset{2}{\mathbf{U}}{}^2\overset{1}{\mathbf{R}}{}^{\mathrm{e}}\overset{1}{\mathbf{R}}{}^{\mathrm{p}}. \tag{19}$$

Similar to the procedure outlined in the previous section we derive a decomposition of the stretch tensor $\overset{20}{\mathbf{U}}$ in terms of elastic and plastic contributions. Defining back-rotated stretch tensors for the second deformation (see Fig. 2)

$$\begin{aligned}\overset{20}{\mathbf{U}}{}^{\mathrm{p}} &:= \overset{1}{\mathbf{R}}{}^{\mathrm{pT}}\overset{1}{\mathbf{R}}{}^{\mathrm{eT}}\overset{2}{\mathbf{U}}{}^{\mathrm{p}}\overset{1}{\mathbf{R}}{}^{\mathrm{e}}\overset{1}{\mathbf{R}}{}^{\mathrm{p}},\\ \overset{20}{\mathbf{U}}{}^{\mathrm{e}} &:= (\overset{1}{\mathbf{R}}{}^{\mathrm{p}})^{\mathrm{T}}(\overset{1}{\mathbf{R}}{}^{\mathrm{e}})^{\mathrm{T}}(\overset{2}{\mathbf{R}}{}^{\mathrm{p}})^{\mathrm{T}}\overset{2}{\mathbf{U}}{}^{\mathrm{e}}\overset{2}{\mathbf{R}}{}^{\mathrm{p}}\overset{1}{\mathbf{R}}{}^{\mathrm{e}}\overset{1}{\mathbf{R}}{}^{\mathrm{p}}\end{aligned} \tag{20}$$

the deformation gradient (18) can be written in the form

$$\overset{20}{\mathbf{F}} = \overset{2}{\mathbf{R}}{}^{\mathrm{e}}\overset{2}{\mathbf{R}}{}^{\mathrm{p}}\overset{1}{\mathbf{R}}{}^{\mathrm{e}}\overset{1}{\mathbf{R}}{}^{\mathrm{p}}\overset{20}{\mathbf{U}}{}^{\mathrm{e}}\overset{20}{\mathbf{U}}{}^{\mathrm{p}}. \tag{21}$$

To the composition $\overset{20}{\mathbf{U}}{}^{\mathrm{e}}\overset{20}{\mathbf{U}}{}^{\mathrm{p}}$, identified with a gradient $\overset{2*}{\mathbf{F}}$, we apply the polar decomposition theorem

$$\overset{2*}{\mathbf{F}} := \overset{20}{\mathbf{U}}{}^{\mathrm{e}}\overset{20}{\mathbf{U}}{}^{\mathrm{p}} = \overset{2*}{\mathbf{R}}\overset{2*}{\mathbf{U}}. \tag{22}$$

Introducing (22) into (21) yields

$$\overset{20}{\mathbf{F}} = \overset{2}{\mathbf{R}}{}^{\mathrm{e}}\overset{2}{\mathbf{R}}{}^{\mathrm{p}}\overset{1}{\mathbf{R}}{}^{\mathrm{e}}\overset{1}{\mathbf{R}}{}^{\mathrm{p}}\overset{2*}{\mathbf{R}}\overset{2*}{\mathbf{U}} = \overset{20}{\mathbf{R}}\overset{20}{\mathbf{U}}, \tag{23}$$

with

$$\overset{20}{\mathbf{U}} = \overset{2*}{\mathbf{U}}, \quad \overset{20}{\mathbf{R}} = \overset{2}{\mathbf{R}}{}^{\mathrm{e}}\overset{2}{\mathbf{R}}{}^{\mathrm{p}}\overset{1}{\mathbf{R}}{}^{\mathrm{e}}\overset{1}{\mathbf{R}}{}^{\mathrm{p}}\overset{2*}{\mathbf{R}}. \tag{24}$$

From (21) and (22) the decomposition of the stretch tensor $\overset{20}{\mathbf{U}}$ into elastic and plastic parts

$$\overset{20}{\mathbf{U}}{}^2 = \overset{20}{\mathbf{U}}{}^{\mathrm{p}}\overset{20}{\mathbf{U}}{}^{\mathrm{e}2}\overset{20}{\mathbf{U}}{}^{\mathrm{p}} \tag{25}$$

and the rotation $\overset{2*}{\mathbf{R}}$

$$\overset{2*}{\mathbf{R}} = \overset{20}{\mathbf{U}}{}^{\mathrm{e}}\overset{20}{\mathbf{U}}{}^{\mathrm{p}}\sqrt{(\overset{20}{\mathbf{U}}{}^{\mathrm{p}})^{-1}(\overset{20}{\mathbf{U}}{}^{\mathrm{e}})^{-2}(\overset{20}{\mathbf{U}}{}^{\mathrm{p}})^{-1}} \tag{26}$$

are obtained.

If we assume that the first deformation is fixed and known by preceding calculations then the configuration $\overset{10}{\mathfrak{B}}{}^{\mathrm{ep}}$ is fixed and known and therefore the stretch tensors $\overset{20}{\mathbf{U}}$ and $\overset{20}{\mathbf{U}}{}^{\mathrm{p}}$ are of Lagrangean type, while $\overset{20}{\mathbf{U}}{}^{\mathrm{e}}$ is not. To obtain a Lagrangean-type elastic stretch tensor denoted by $\overset{2*}{\mathbf{U}}{}^{\mathrm{e}}$ and orientationally referred to $\overset{10}{\mathfrak{B}}{}^{\mathrm{ep}}$ we define

$$\overset{2*}{\mathbf{U}}{}^{\mathrm{e}} := \overset{2*}{\mathbf{R}}{}^{\mathrm{T}}\overset{20}{\mathbf{U}}{}^{\mathrm{e}}\overset{2*}{\mathbf{R}}, \quad \overset{2*}{\mathbf{U}}{}^{\mathrm{p}} := \overset{2*}{\mathbf{R}}{}^{\mathrm{T}}\overset{20}{\mathbf{U}}{}^{\mathrm{p}}\overset{2*}{\mathbf{R}}. \tag{27}$$

Then, analogously to (14), eqns (21), (22) lead, with (27), to

$$\begin{aligned}\overset{20}{\mathbf{F}} &= \overset{2}{\mathbf{R}}{}^{\mathrm{e}}\overset{2}{\mathbf{R}}{}^{\mathrm{p}}\overset{1}{\mathbf{R}}{}^{\mathrm{e}}\overset{1}{\mathbf{R}}{}^{\mathrm{p}}\overset{2*}{\mathbf{R}}{}^2\overset{2*}{\mathbf{U}}{}^{\mathrm{p}}\overset{2*}{\mathbf{U}}{}^{\mathrm{e}},\\ \overset{20}{\mathbf{U}}{}^2 &= \overset{2*}{\mathbf{U}}{}^{\mathrm{e}}(\overset{2*}{\mathbf{U}}{}^{\mathrm{p}})^2\overset{2*}{\mathbf{U}}{}^{\mathrm{e}},\end{aligned} \tag{28}$$

where $(28)_2$ represents a decomposition of the stretch tensor $\overset{20}{\mathbf{U}}$ into elastic and plastic parts, as an alternative to decomposition (25).

2.3. *Total elastic–plastic deformation*

In the previous section we considered stretches and rotations of the second superposed deformation. To construct an efficient numerical approximation procedure allowing the analysis of finite elastic–plastic deformation of structures the magnitude of the second deformation must be restricted with respect to some norm, as will be shown in the next section. Then from time to time an update has to be performed and the total elastic and plastic stretches and also the orientational change of the reference configuration must be determined.

Defining for the first deformation

$$\overset{10}{\mathbf{U}}{}^{\mathrm{e}} := (\overset{1}{\mathbf{R}}{}^{\mathrm{p}})^{\mathrm{T}}\, \overset{1}{\mathbf{U}}{}^{\mathrm{e}}\, \overset{1}{\mathbf{R}}{}^{\mathrm{p}} \tag{29}$$

we can derive the following formulae for the total deformation gradient

$$\mathbf{F} = \overset{2}{\mathbf{F}}\,\overset{1}{\mathbf{F}} = \overset{20}{\mathbf{F}}\,(\overset{1}{\mathbf{R}}{}^{\mathrm{p}})^{\mathrm{T}}\,(\overset{1}{\mathbf{R}}{}^{\mathrm{e}})^{\mathrm{T}}\,\overset{1}{\mathbf{F}} = \overset{2}{\mathbf{R}}{}^{\mathrm{e}}\,\overset{2}{\mathbf{R}}{}^{\mathrm{p}}\,\overset{1}{\mathbf{R}}{}^{\mathrm{e}}\,\overset{1}{\mathbf{R}}{}^{\mathrm{p}}\,\overset{20}{\mathbf{U}}{}^{\mathrm{e}}\,\overset{20}{\mathbf{U}}{}^{\mathrm{p}}\,\overset{10}{\mathbf{U}}{}^{\mathrm{e}}\,\overset{1}{\mathbf{U}}{}^{\mathrm{p}} \tag{30}$$

and

$$\mathbf{F} = \overset{2}{\mathbf{R}}{}^{\mathrm{e}}\,\overset{2}{\mathbf{R}}{}^{\mathrm{p}}\,\overset{1}{\mathbf{R}}{}^{\mathrm{e}}\,\overset{1}{\mathbf{R}}{}^{\mathrm{p}}\,\overset{2*}{\mathbf{R}}{}^{2}\,\overset{2*}{\mathbf{U}}{}^{\mathrm{p}}\,\overset{2*}{\mathbf{U}}{}^{\mathrm{e}}\,\overset{10}{\mathbf{U}}{}^{\mathrm{e}}\,\overset{1}{\mathbf{U}}{}^{\mathrm{p}}, \tag{31}$$

yielding the total stretch decompositions

$$\mathbf{U}^2 = \mathbf{F}^{\mathrm{T}}\mathbf{F} = \overset{1}{\mathbf{U}}{}^{\mathrm{p}}\,\overset{10}{\mathbf{U}}{}^{\mathrm{e}}\,\overset{20}{\mathbf{U}}{}^{\mathrm{p}}\,(\overset{20}{\mathbf{U}}{}^{\mathrm{e}})^2\,\overset{20}{\mathbf{U}}{}^{\mathrm{p}}\,\overset{10}{\mathbf{U}}{}^{\mathrm{e}}\,\overset{1}{\mathbf{U}}{}^{\mathrm{p}}, \tag{32}$$

$$\mathbf{U}^2 = \overset{1}{\mathbf{U}}{}^{\mathrm{p}}\,\overset{10}{\mathbf{U}}{}^{\mathrm{e}}\,\overset{2*}{\mathbf{U}}{}^{\mathrm{e}}\,(\overset{2*}{\mathbf{U}}{}^{\mathrm{p}})^2\,\overset{2*}{\mathbf{U}}{}^{\mathrm{e}}\,\overset{10}{\mathbf{U}}{}^{\mathrm{e}}\,\overset{1}{\mathbf{U}}{}^{\mathrm{p}} \tag{33}$$

and

$$\overset{20}{\mathbf{U}}{}^2 = (\overset{10}{\mathbf{U}}{}^{\mathrm{e}})^{-1}\,(\overset{1}{\mathbf{U}}{}^{\mathrm{p}})^{-1}\,\mathbf{U}^2\,(\overset{1}{\mathbf{U}}{}^{\mathrm{p}})^{-1}\,(\overset{10}{\mathbf{U}}{}^{\mathrm{e}})^{-1}. \tag{34}$$

As can be seen from the total deformation gradient $\mathbf{F}$ according to (30) and (31) elastic and plastic stretches appear separately for the first and second deformations. Our intention is now to determine total elastic and total plastic stretches. This can be performed by using a similar procedure to that outlined above.

First we identify in eqn (31) the composition $\overset{2*}{\mathbf{U}}{}^{\mathrm{p}}\,\overset{2*}{\mathbf{U}}{}^{\mathrm{e}}\,\overset{10}{\mathbf{U}}{}^{\mathrm{e}}$ with a deformation gradient $\bar{\mathbf{F}}$ and apply the polar decomposition theorem,

$$\bar{\mathbf{F}} := \overset{2*}{\mathbf{U}}{}^{\mathrm{p}}\,\overset{2*}{\mathbf{U}}{}^{\mathrm{e}}\,\overset{10}{\mathbf{U}}{}^{\mathrm{e}} = \bar{\mathbf{R}}\bar{\mathbf{U}} \tag{35}$$

leading to

$$\bar{\mathbf{R}} = \overset{2*}{\mathbf{U}}{}^{\mathrm{p}}\,\overset{2*}{\mathbf{U}}{}^{\mathrm{e}}\,\overset{10}{\mathbf{U}}{}^{\mathrm{e}}\,\bar{\mathbf{U}}^{-1}, \quad \bar{\mathbf{U}}^2 = \overset{10}{\mathbf{U}}{}^{\mathrm{e}}\,\overset{2*}{\mathbf{U}}{}^{\mathrm{e}}\,(\overset{2*}{\mathbf{U}}{}^{\mathrm{p}})^2\,\overset{2*}{\mathbf{U}}{}^{\mathrm{e}}\,\overset{10}{\mathbf{U}}{}^{\mathrm{e}}. \tag{36}$$

Then we transform (35) and (31) as follows:

$$\bar{\mathbf{F}} = \bar{\mathbf{R}}\bar{\mathbf{F}}^{\mathrm{T}}\bar{\mathbf{R}} = \bar{\mathbf{R}}\,\overset{10}{\mathbf{U}}{}^{\mathrm{e}}\,\overset{2*}{\mathbf{U}}{}^{\mathrm{e}}\,\overset{2*}{\mathbf{U}}{}^{\mathrm{p}}\,\bar{\mathbf{R}}, \tag{37}$$

$$\mathbf{F} = \overset{2}{\mathbf{R}}{}^{\mathrm{e}}\,\overset{2}{\mathbf{R}}{}^{\mathrm{p}}\,\overset{1}{\mathbf{R}}{}^{\mathrm{e}}\,\overset{1}{\mathbf{R}}{}^{\mathrm{p}}\,\overset{2*}{\mathbf{R}}{}^{2}\,\bar{\mathbf{R}}\,\overset{10}{\mathbf{U}}{}^{\mathrm{e}}\,\overset{2*}{\mathbf{U}}{}^{\mathrm{e}}\,\overset{2*}{\mathbf{U}}{}^{\mathrm{p}}\,\bar{\mathbf{R}}\,\overset{1}{\mathbf{U}}{}^{\mathrm{p}}. \tag{38}$$

To derive the total elastic and total plastic stretches we consider in (38) the compositions

$\overset{10}{\mathbf{U}}{}^{e}\overset{2*}{\mathbf{U}}{}^{e}$ and $\overset{2*}{\mathbf{U}}{}^{p}\bar{\mathbf{R}}\overset{1}{\mathbf{U}}{}^{p}$ and apply again the polar decomposition theorem:

$$\begin{aligned}&\overset{+}{\mathbf{F}}{}^{e} := \overset{10}{\mathbf{U}}{}^{e}\overset{2*}{\mathbf{U}}{}^{e} = \overset{+}{\mathbf{R}}{}^{e}\overset{+}{\mathbf{U}}{}^{e},\\ &\overset{+}{\mathbf{R}}{}^{e} = \overset{10}{\mathbf{U}}{}^{e}\overset{2*}{\mathbf{U}}{}^{e}(\overset{+}{\mathbf{U}}{}^{e})^{-1},\quad (\overset{+}{\mathbf{U}}{}^{e})^{2} = \overset{2*}{\mathbf{U}}{}^{e}(\overset{10}{\mathbf{U}}{}^{e})^{2}\overset{2*}{\mathbf{U}}{}^{e}\end{aligned} \tag{39}$$

and

$$\begin{aligned}&\overset{+}{\mathbf{F}}{}^{p} := \overset{2*}{\mathbf{U}}{}^{p}\bar{\mathbf{R}}\overset{1}{\mathbf{U}}{}^{p} = \overset{+}{\mathbf{R}}{}^{p}\overset{+}{\mathbf{U}}{}^{p},\quad \overset{+}{\mathbf{U}}{}^{p} = \mathbf{U}^{p},\\ &\overset{+}{\mathbf{R}}{}^{p} = \overset{2*}{\mathbf{U}}{}^{p}\bar{\mathbf{R}}\overset{1}{\mathbf{U}}{}^{p}(\mathbf{U}^{p})^{-1},\quad (\mathbf{U}^{p})^{2} = \overset{1}{\mathbf{U}}{}^{p}\bar{\mathbf{R}}^{T}(\overset{2*}{\mathbf{U}}{}^{p})^{2}\bar{\mathbf{R}}\overset{1}{\mathbf{U}}{}^{p}.\end{aligned} \tag{40}$$

With (39), (40) and with a back-rotated total elastic stretch tensor

$$\mathring{\mathbf{U}}{}^{e} := (\overset{+}{\mathbf{R}}{}^{p})^{T}\overset{+}{\mathbf{U}}{}^{e}\overset{+}{\mathbf{R}}{}^{p} \tag{41}$$

the total deformation gradient (38) is obtained in the final form

$$\mathbf{F} = \overset{2}{\mathbf{R}}{}^{e}\overset{2}{\mathbf{R}}{}^{p}\overset{1}{\mathbf{R}}{}^{e}\overset{1}{\mathbf{R}}{}^{p}\overset{2*}{\mathbf{R}}{}^{2}\bar{\mathbf{R}}\overset{+}{\mathbf{R}}{}^{e}\overset{+}{\mathbf{R}}{}^{p}\mathring{\mathbf{U}}{}^{e}\mathbf{U}^{p}:\quad T\mathring{\mathfrak{B}} \rightarrow T\overset{2}{\mathfrak{B}} \tag{42}$$

leading to the following decomposition of the total stretch $\mathbf{U}$ into total elastic and total plastic stretches $\mathring{\mathbf{U}}{}^{e}$ and $\mathbf{U}^{p}$:

$$\mathbf{U}^{2} = \mathbf{F}^{T}\mathbf{F} = \mathbf{U}^{p}(\mathring{\mathbf{U}}{}^{e})^{2}\mathbf{U}^{p}. \tag{43}$$

The tensors $\mathbf{U}$ and $\mathbf{U}^{p}$ are referred to the undeformed configuration $\mathring{\mathfrak{B}}$, while $\mathring{\mathbf{U}}{}^{e}$ is directionally referred to the configuration $\mathring{\mathfrak{B}}{}^{ep}$.

For our further considerations it is essential to determine the orientational change between the configurations $\overset{10}{\mathfrak{B}}{}^{ep}$, reference configuration for the superposed second deformation, as outlined in the previous section, and $\mathring{\mathfrak{B}}{}^{ep}$ reference configuration for future superposed back-rotated stretches, as indicated above. Denoting this rotational change by a rotation tensor $\overset{\#}{\mathbf{R}}$ a comparison of (18), (23) with (42) leads to the result:

$$\overset{\#}{\mathbf{R}} = \overset{2*}{\mathbf{R}}\bar{\mathbf{R}}\overset{+}{\mathbf{R}}{}^{e}\overset{+}{\mathbf{R}}{}^{p} = \mathring{\mathbf{U}}{}^{e}\mathbf{U}^{p}(\overset{1}{\mathbf{U}}{}^{p})^{-1}(\overset{10}{\mathbf{U}}{}^{e})^{-1}\overset{20}{\mathbf{U}}{}^{-1}. \tag{44}$$

Rotation (44) will be needed in the numerical approximation procedure, when an update of the first deformation has to be performed.

3. LOGARITHMIC STRAINS AND RATES OF LOGARITHMIC STRAINS FOR THE SUPERPOSED DEFORMATION

3.1. *Additivity of the logarithmic strains for moderately large strains*

To perform the analysis of deformations undergoing finite elastic and finite plastic strains we decompose the total deformation into two parts assuming that the first deformation is known from preceding calculations. For the second deformation we presume that the associated strains are moderately large. When during the computational analysis the second deformation reaches the limit of moderate strains (e.g. about 10%) then the first deformation has to be updated ensuring that the strains of the second deformation are restricted to being only moderately large, while the total strains may become arbitrarily large. With the notation of Section 2.2 we assume

$$\begin{aligned}\|\overset{20}{\mathbf{E}}\| &= \|\tfrac{1}{2}(\overset{20}{\mathbf{U}}{}^{2}-\mathbf{1})\| = 0(\theta),\\ \|\overset{2*}{\mathbf{E}}{}^{e}\| &= \|\tfrac{1}{2}(\overset{2*}{\mathbf{U}}{}^{e})^{2}-\mathbf{1})\| = 0(\theta),\\ \|\overset{2*}{\mathbf{E}}{}^{p}\| &= \|\tfrac{1}{2}(\overset{2*}{\mathbf{U}}{}^{p})^{2}-\mathbf{1})\| = 0(\theta),\end{aligned} \tag{45}$$

with

$$\|\mathbf{0}(\theta^2)\| = 0(\theta^2) \ll 1, \tag{46}$$

where $\mathbf{0}(\theta^2)$ is a small, tensor-valued quantity. Estimations of the type (45), (46) are frequently applied in the derivation of shell theories [e.g. Pietraszkiewicz (1977, 1989), Nolte and Stumpf (1983), Schmidt and Stumpf (1984), and Schieck *et al.* (1992)].

After analysing the first deformation the stretch tensors $\overset{1}{\mathbf{U}}{}^p$ and $\overset{10}{\mathbf{U}}{}^e$ are known and the total stretch tensor $\mathbf{U}^2$ can be computed from the total deformation gradient $\mathbf{F}$.

The back-rotated stretch tensor $\overset{20}{\mathbf{U}}$ of the second deformation follows from eqn (34) with

$$\overset{20}{\mathbf{U}}{}^2 = (\mathbf{1}+2\overset{20}{\mathbf{E}}) = (\overset{10}{\mathbf{U}}{}^e)^{-1}(\overset{1}{\mathbf{U}}{}^p)^{-1}\mathbf{U}^2(\overset{1}{\mathbf{U}}{}^p)^{-1}(\overset{10}{\mathbf{U}}{}^e)^{-1}. \tag{47}$$

Using the estimation $(45)_1$ the Lagrangean logarithmic strain tensor $\overset{20}{\mathbf{H}}$ of the second deformation can be defined and approximated by series expansion as

$$\overset{20}{\mathbf{H}} := \ln \overset{20}{\mathbf{U}} = \tfrac{1}{2}\ln(\mathbf{1}+2\overset{20}{\mathbf{E}}) = (\overset{20}{\mathbf{E}}-\overset{20}{\mathbf{E}}{}^2)(\mathbf{1}+\mathbf{0}(\theta^2)). \tag{48}$$

Analogously to (48) we define the logarithmic elastic and plastic strain tensors of the second deformation approximated by

$$\overset{20}{\mathbf{H}}{}^e := \ln \overset{20}{\mathbf{U}}{}^e = (\overset{20}{\mathbf{E}}{}^e-(\overset{20}{\mathbf{E}}{}^e)^2)(\mathbf{1}+\mathbf{0}(\theta^2)), \tag{49}$$

$$\overset{20}{\mathbf{H}}{}^p := \ln \overset{20}{\mathbf{U}}{}^p = (\overset{20}{\mathbf{E}}{}^p-(\overset{20}{\mathbf{E}}{}^p)^2)(\mathbf{1}+\mathbf{0}(\theta^2)), \tag{50}$$

$$\overset{2*}{\mathbf{H}}{}^e := \ln \overset{2*}{\mathbf{U}}{}^e = (\overset{2*}{\mathbf{E}}{}^e-(\overset{2*}{\mathbf{E}}{}^e)^2)(\mathbf{1}+\mathbf{0}(\theta^2)), \tag{51}$$

$$\overset{2*}{\mathbf{H}}{}^p := \ln \overset{2*}{\mathbf{U}}{}^p = (\overset{2*}{\mathbf{E}}{}^p-(\overset{2*}{\mathbf{E}}{}^p)^2)(\mathbf{1}+\mathbf{0}(\theta^2)). \tag{52}$$

Using a series expansion, $\overset{20}{\mathbf{U}}{}^p$ can be approximated as

$$\overset{20}{\mathbf{U}}{}^p = \sqrt{\mathbf{1}+2\overset{20}{\mathbf{E}}{}^p} = (\mathbf{1}+\overset{20}{\mathbf{E}}{}^p-\tfrac{1}{2}(\overset{20}{\mathbf{E}}{}^p)^2)(\mathbf{1}+\mathbf{0}(\theta^3)). \tag{53}$$

Applying this estimation technique to the total stretch tensor $\overset{20}{\mathbf{U}}$ of the second deformation according to (25) we obtain

$$\begin{aligned}
\overset{20}{\mathbf{U}}{}^2 = \mathbf{1}+2\overset{20}{\mathbf{E}} &= \overset{20}{\mathbf{U}}{}^p(\overset{20}{\mathbf{U}}{}^e)^2\overset{20}{\mathbf{U}}{}^p \\
&= \sqrt{\mathbf{1}+2\overset{20}{\mathbf{E}}{}^p}(\mathbf{1}+2\overset{20}{\mathbf{E}}{}^e)\sqrt{\mathbf{1}+2\overset{20}{\mathbf{E}}{}^p} \\
&= (\mathbf{1}+\overset{20}{\mathbf{E}}{}^p-\tfrac{1}{2}(\overset{20}{\mathbf{E}}{}^p)^2)(\mathbf{1}+2\overset{20}{\mathbf{E}}{}^e)(\mathbf{1}+\overset{20}{\mathbf{E}}{}^p-\tfrac{1}{2}(\overset{20}{\mathbf{E}}{}^p)^2)(\mathbf{1}+\mathbf{0}(\theta^3)) \\
&= \mathbf{1}+2(\overset{20}{\mathbf{E}}{}^e+\overset{20}{\mathbf{E}}{}^p+\overset{20}{\mathbf{E}}{}^e\overset{20}{\mathbf{E}}{}^p+\overset{20}{\mathbf{E}}{}^p\overset{20}{\mathbf{E}}{}^e)(\mathbf{1}+\mathbf{0}(\theta^2)).
\end{aligned} \tag{54}$$

With (54) we derive the following form of the Lagrangean logarithmic (Hencky) strain tensor (48)

$$\overset{20}{\mathbf{H}} = \tfrac{1}{2}\ln \overset{20}{\mathbf{U}}{}^2 = (\overset{20}{\mathbf{E}} - \overset{20}{\mathbf{E}}{}^2)(\mathbf{1}+\mathbf{0}(\theta^2))$$
$$= [(\overset{20}{\mathbf{E}}{}^{\mathrm{e}} + \overset{20}{\mathbf{E}}{}^{\mathrm{p}} + \overset{20}{\mathbf{E}}{}^{\mathrm{e}}\overset{20}{\mathbf{E}}{}^{\mathrm{p}} + \overset{20}{\mathbf{E}}{}^{\mathrm{p}}\overset{20}{\mathbf{E}}{}^{\mathrm{e}})$$
$$- (\overset{20}{\mathbf{E}}{}^{\mathrm{e}} + \overset{20}{\mathbf{E}}{}^{\mathrm{p}} + \overset{20}{\mathbf{E}}{}^{\mathrm{e}}\overset{20}{\mathbf{E}}{}^{\mathrm{p}} + \overset{20}{\mathbf{E}}{}^{\mathrm{p}}\overset{20}{\mathbf{E}}{}^{\mathrm{e}})^2](\mathbf{1}+\mathbf{0}(\theta^2))$$
$$= [(\overset{20}{\mathbf{E}}{}^{\mathrm{e}} - (\overset{20}{\mathbf{E}}{}^{\mathrm{e}})^2) + (\overset{20}{\mathbf{E}}{}^{\mathrm{p}} - (\overset{20}{\mathbf{E}}{}^{\mathrm{p}})^2)](\mathbf{1}+\mathbf{0}(\theta^2)), \tag{55}$$

leading to the additive decomposition:

$$\overset{20}{\mathbf{H}} = (\overset{20}{\mathbf{H}}{}^{\mathrm{e}} + \overset{20}{\mathbf{H}}{}^{\mathrm{p}})(\mathbf{1}+\mathbf{0}(\theta^2)). \tag{56}$$

A second additive decomposition of $\overset{20}{\mathbf{H}}$ can be obtained by applying formula (28)

$$\overset{20}{\mathbf{H}} = (\overset{2*}{\mathbf{H}}{}^{\mathrm{e}} + \overset{2*}{\mathbf{H}}{}^{\mathrm{p}})(\mathbf{1}+\mathbf{0}(\theta^2)). \tag{57}$$

A comparison of (56) and (57) shows that for moderate strains of the superposed deformation the tensors $\overset{2*}{\mathbf{U}}{}^{\mathrm{e}}$ and $\overset{2*}{\mathbf{U}}{}^{\mathrm{p}}$ can be approximated by $\overset{20}{\mathbf{U}}{}^{\mathrm{e}}$ and $\overset{20}{\mathbf{U}}{}^{\mathrm{p}}$. Therefore besides the tensors $\overset{2*}{\mathbf{U}}{}^{\mathrm{e}}$ and $\overset{20}{\mathbf{U}}{}^{\mathrm{p}}$ referred to the fixed configuration $\overset{10}{\mathfrak{B}}{}^{\mathrm{ep}}$ given by the first deformation the tensors $\overset{20}{\mathbf{U}}{}^{\mathrm{e}}$ and $\overset{2*}{\mathbf{U}}{}^{\mathrm{p}}$ are also referred to the same configuration within the error margin. The same statement is valid for the elastic and plastic logarithmic strain tensors.

It should be pointed out that eqns (56) and (57) present an additive decomposition of the total Lagrangean logarithmic strain of the second deformation into purely elastic and purely plastic contributions, in contrast to the Green and Almansi strain tensors, which in the general case of large elastic and plastic strains cannot be decomposed additively into purely elastic and plastic parts [see e.g. Stumpf and Badur (1990) and Stumpf (1991, 1993)].

3.2. *Additivity of the rates of the logarithmic strain tensors*

The stretch of the second deformation is determined by eqn (47), where the stretches $\overset{10}{\mathbf{U}}{}^{\mathrm{e}}$ and $\overset{1}{\mathbf{U}}{}^{\mathrm{p}}$ are known from preceding calculations. The stretch tensor $\overset{20}{\mathbf{U}}$ and its logarithm $\overset{20}{\mathbf{H}}$ according to (48) are referred to the fixed configuration $\overset{10}{\mathfrak{B}}{}^{\mathrm{ep}}$. Therefore the variation $\delta\overset{20}{\mathbf{E}}$ and the second variation $\Delta\delta\overset{20}{\mathbf{E}}$ are objective and can be derived in the usual way leading to

$$\delta\overset{20}{\mathbf{E}} = \tfrac{1}{2}\delta\overset{20}{\mathbf{U}}{}^2 = (\overset{10}{\mathbf{U}}{}^{\mathrm{e}})^{-1}(\overset{1}{\mathbf{U}}{}^{\mathrm{p}})^{-1}\delta\mathbf{E}(\overset{1}{\mathbf{U}}{}^{\mathrm{p}})^{-1}(\overset{10}{\mathbf{U}}{}^{\mathrm{e}})^{-1} \tag{58}$$

and

$$\Delta\delta\overset{20}{\mathbf{E}} = (\overset{10}{\mathbf{U}}{}^{\mathrm{e}})^{-1}(\overset{1}{\mathbf{U}}{}^{\mathrm{p}})^{-1}\Delta\delta\mathbf{E}(\overset{1}{\mathbf{U}}{}^{\mathrm{p}})^{-1}(\overset{10}{\mathbf{U}}{}^{\mathrm{e}})^{-1}, \tag{59}$$

with

$$\delta\mathbf{E} = \tfrac{1}{2}(\delta\mathbf{F}^{\mathrm{T}}\mathbf{F} + \mathbf{F}^{\mathrm{T}}\delta\mathbf{F}) \tag{60}$$

and

$$\Delta\delta\mathbf{E} = \delta\Delta\mathbf{E} = \tfrac{1}{2}(\delta\mathbf{F}^{\mathrm{T}}\Delta\mathbf{F} + \Delta\mathbf{F}^{\mathrm{T}}\delta\mathbf{F}). \tag{61}$$

Here $\delta(\cdot)$ and $\Delta\delta(\cdot)$ indicate the first and second variation, respectively.

The first variation of the logarithmic strain tensor $\overset{20}{\mathbf{H}}$ (48) can be approximated by

$$\delta\overset{20}{\mathbf{H}} = (\delta\overset{20}{\mathbf{E}} - \overset{20}{\mathbf{E}}\,\delta\overset{20}{\mathbf{E}} - \delta\overset{20}{\mathbf{E}}\,\overset{20}{\mathbf{E}})(\mathbf{1}+\mathbf{0}(\theta^2)) \tag{62}$$

and the second variation by

$$\Delta\delta\overset{20}{\mathbf{H}} = (\Delta\delta\overset{20}{\mathbf{E}} - \Delta\overset{20}{\mathbf{E}}\,\delta\overset{20}{\mathbf{E}} - \delta\overset{20}{\mathbf{E}}\,\Delta\overset{20}{\mathbf{E}} - \overset{20}{\mathbf{E}}\,\Delta\delta\overset{20}{\mathbf{E}} - \Delta\delta\overset{20}{\mathbf{E}}\,\overset{20}{\mathbf{E}})(\mathbf{1}+\mathbf{0}(\theta^2)). \tag{63}$$

Analogous results are also obtained for the first and second variation of the logarithmic strains $\overset{2*}{\mathbf{H}}{}^e$ and $\overset{2*}{\mathbf{H}}{}^p$.

Within the moderate strain approximation for the superposed deformation we can decompose the logarithmic strain tensor $\overset{20}{\mathbf{H}}$ additively into its elastic and plastic parts, as shown in eqns (56) and (57). Because the elastic and plastic logarithmic strain tensors are referred to the fixed configuration $\overset{10}{\mathfrak{B}}{}^{ep}$ within this approximation also the first and second variation (62) and (63) can be decomposed additively leading to

$$\delta\overset{20}{\mathbf{H}} = (\delta\overset{2*}{\mathbf{H}}{}^e+\delta\overset{2*}{\mathbf{H}}{}^p)(\mathbf{1}+\mathbf{0}(\theta^2)) \tag{64}$$

and

$$\Delta\delta\overset{20}{\mathbf{H}} = (\Delta\delta\overset{2*}{\mathbf{H}}{}^e+\Delta\delta\overset{2*}{\mathbf{H}}{}^p)(\mathbf{1}+\mathbf{0}(\theta^2)). \tag{65}$$

In order to define in the next section the stress tensor, which is work-conjugate to $\delta\overset{20}{\mathbf{H}}$, we present here the geometrical interpretation of $\delta\overset{20}{\mathbf{H}}$. Within the error margin eqn (62) can be rewritten as

$$\begin{aligned}\delta\overset{20}{\mathbf{H}} &= [(\mathbf{1}-\overset{20}{\mathbf{E}})\,\delta\overset{20}{\mathbf{E}}\,(\mathbf{1}-\overset{20}{\mathbf{E}})](\mathbf{1}+\mathbf{0}(\theta^2))\\ &= (\overset{20}{\mathbf{U}}{}^{-1}\,\delta\overset{20}{\mathbf{E}}\,\overset{20}{\mathbf{U}}{}^{-1})(\mathbf{1}+\mathbf{0}(\theta^2))\\ &= (\overset{20}{\mathbf{U}}{}^{-T}\,\delta\overset{20}{\mathbf{E}}\,\overset{20}{\mathbf{U}}{}^{-1})(\mathbf{1}+\mathbf{0}(\theta^2)).\end{aligned} \tag{66}$$

The polar decomposition (23) of the deformation gradient $\overset{20}{\mathbf{F}}: T\overset{10}{\mathfrak{B}}{}^{ep} \to T\overset{2}{\mathfrak{B}}$ yields

$$\overset{20}{\mathbf{U}}{}^{-1} = \overset{20}{\mathbf{F}}{}^{-1}\overset{20}{\mathbf{R}}, \quad \overset{20}{\mathbf{U}}{}^{-T} = \overset{20}{\mathbf{R}}{}^{T}\overset{20}{\mathbf{F}}{}^{-T}. \tag{67}$$

Introducing (67) into (66) and defining the objective Lie-variation $\delta_{\mathfrak{L}}(\cdot) := \mathfrak{L}(\cdot)\,\delta t$, where $\mathfrak{L}(\cdot)$ is the Lie derivative, we obtain

$$\delta\overset{20}{\mathbf{H}} = \overset{20}{\mathbf{R}}{}^{T}\overset{20}{\mathbf{F}}{}^{-T}\delta\overset{20}{\mathbf{E}}\,\overset{20}{\mathbf{F}}{}^{-1}\overset{20}{\mathbf{R}}(\mathbf{1}+\mathbf{0}(\theta^2)) = \overset{20}{\mathbf{R}}{}^{T}\delta_{\mathfrak{L}}(\overset{20}{\mathbf{e}})\,\overset{20}{\mathbf{R}}(\mathbf{1}+\mathbf{0}(\theta^2)) \tag{68}$$

with the Almansi-type strain tensor $\overset{20}{\mathbf{e}}$

$$\overset{20}{\mathbf{e}} := \overset{20}{\mathbf{F}}{}^{-T}\overset{20}{\mathbf{E}}\,\overset{20}{\mathbf{F}}{}^{-1}. \tag{69}$$

It can be shown that for a constant first deformation

$$\delta_{\mathfrak{L}}(\overset{20}{\mathbf{e}}) = \delta_{\mathfrak{L}}(\mathbf{e}), \tag{70}$$

where $\mathbf{e}$ is the total Almansi strain tensor. It follows that $\delta\overset{20}{\mathbf{H}}$ is the rotational pull-back with $\overset{20}{\mathbf{R}}$ of the objective Lie-variation $\delta_{\mathfrak{L}}\mathbf{e}$.

To formulate the constitutive equations in Section 4 we have to consider the elastic logarithmic strain tensor $\overset{+}{\mathbf{H}}{}^e := \ln\overset{+}{\mathbf{U}}{}^e$ with $\overset{+}{\mathbf{U}}{}^e$ according to eqn $(39)_3$

$$\overset{+}{\mathbf{H}}{}^e := \tfrac{1}{2}\ln(\overset{+}{\mathbf{U}}{}^e)^2 = \tfrac{1}{2}\ln(\overset{2*}{\mathbf{U}}{}^e(\overset{10}{\mathbf{U}}{}^e)^2\overset{2*}{\mathbf{U}}{}^e), \tag{71}$$

and to determine its first and second variation.

We assume that the elastic stretch tensor $\overset{10}{\mathbf{U}}{}^e$ is determined by preceding calculations. Then $\overset{+}{\mathbf{H}}{}^e$ depends on $\overset{2*}{\mathbf{U}}{}^e$ and $\overset{2*}{\mathbf{H}}{}^e = \ln\overset{2*}{\mathbf{U}}{}^e$, respectively. We derive the first variation in the following form

$$\overset{+}{\delta\mathbf{H}^e} = \overset{+}{\delta\mathbf{H}^e}(\overset{10}{\mathbf{U}^e}, \overset{2*}{\mathbf{H}^e}; \overset{2*}{\delta\mathbf{H}^e})$$
$$= \lim_{t\to 0}\frac{1}{2t}\{\ln[\exp(\overset{2*}{\mathbf{H}^e}+t\overset{2*}{\delta\mathbf{H}^e})(\overset{10}{\mathbf{U}^e})^2\exp(\overset{2*}{\mathbf{H}^e}+t\overset{2*}{\delta\mathbf{H}^e})]$$
$$-\ln[\exp(\overset{2*}{\mathbf{H}^e})(\overset{10}{\mathbf{U}^e})^2\exp(\overset{2*}{\mathbf{H}^e})]\}, \tag{72}$$

where (72) can be obtained as Gâteaux differential

$$\overset{+}{\delta\mathbf{H}^e} = \frac{\overset{+}{\delta\mathbf{H}^e}(\overset{10}{\mathbf{U}^e}, \overset{2*}{\mathbf{H}^e})}{\partial\overset{2*}{\mathbf{H}^e}}\cdot\overset{2*}{\delta\mathbf{H}^e} \tag{73}$$

with the fourth-order tensor $\overset{+}{\delta\mathbf{H}^e}/\partial\overset{2*}{\mathbf{H}^e}$ not presented here because of lack of space.

Analogously we determine the second variation as the second Gâteaux differential

$$\Delta\overset{+}{\delta\mathbf{H}^e} = \Delta\overset{*}{\mathbf{H}^e}\cdot\frac{\partial^2\overset{+}{\mathbf{H}^e}}{\partial\overset{2*}{\mathbf{H}^e}\otimes\partial\overset{2*}{\mathbf{H}^e}}\cdot\overset{2*}{\delta\mathbf{H}^e}. \tag{74}$$

In the special case of total elastic strains remaining within the limit of moderate order such that $\overset{10}{\mathbf{U}^e}$ is not updated keeping always the value $\overset{10}{\mathbf{U}^e} = \mathbf{1}$, then $\overset{+}{\delta\mathbf{H}^e}/\overset{2*}{\delta\mathbf{H}^e} = \mathbf{1}$ and $\partial^2\overset{+}{\mathbf{H}^e}/\partial\overset{2*}{\mathbf{H}^e}\otimes\partial\overset{2*}{\mathbf{H}^e} = \mathbf{0}$. This is also the case, if $\overset{10}{\mathbf{U}}$, $\overset{2*}{\mathbf{H}^e}$, $\overset{2*}{\delta\mathbf{H}^e}$ and $\Delta\overset{2*}{\mathbf{H}^e}$ are coaxial having the same eigendirections.

4. CONSTITUTIVE EQUATIONS FOR LARGE STRAINS

4.1. *Constitutive equations for isotropically-hyperelastic materials*

The variation of the strain rate is given by $\delta\overset{20}{\mathbf{H}}$ (62) under the assumption, that the first deformation is known with fixed $\overset{1}{\mathbf{U}^p}$ and $\overset{10}{\mathbf{U}^e}$. Since $\delta\overset{20}{\mathbf{H}}$ is approximately equal to the Lie derivative of the back-rotated Almansi strain tensor according to (68) and (70) the stress tensor work-conjugate to $\delta\overset{20}{\mathbf{H}}$ is the Kirchhoff stress tensor $\boldsymbol{\tau}$ back-rotated with $\overset{20}{\mathbf{R}}$,

$$\overset{20}{\boldsymbol{\tau}} := \overset{20}{\mathbf{R}}{}^{\mathrm{T}}\boldsymbol{\tau}\overset{20}{\mathbf{R}}. \tag{75}$$

Then the hyperelastic constitutive law can be formulated as follows:

$$\overset{20}{\boldsymbol{\tau}} = \frac{\partial W}{\partial\overset{20}{\mathbf{H}}}, \tag{76}$$

where W denotes the free energy per initial (undeformed) unit volume. The stress tensor $\overset{20}{\boldsymbol{\tau}}$ and the logarithmic strain tensor $\overset{20}{\mathbf{H}}$ are directionally referred to the configuration $\overset{10}{\mathfrak{B}}{}^{ep}$ given by the known first deformation.

The free energy W depends on the total elastic strain and on the temperature T, where the influence of T will not be taken into account in this paper. The total elastic strain may be expressed by the (exactly computed) logarithm of the total elastic stretch tensor $\mathring{\mathbf{U}}^e$, which is given by eqn (41). Thus we can write

$$W = W(\mathring{\mathbf{H}}^e, T), \tag{77}$$

with

$$\overset{\circ}{\mathbf{H}}^{e} := \ln \overset{\circ}{\mathbf{U}}^{e}. \tag{78}$$

Assuming that the elastic part of the material behavior is isotropic, $\overset{\circ}{\mathbf{H}}^{e}$ in (77) may be replaced by $\overset{+}{\mathbf{H}}^{e} = \ln \overset{+}{\mathbf{U}}^{e}$ due to eqn (71), because $\overset{\circ}{\mathbf{U}}^{e}$ and $\overset{+}{\mathbf{U}}^{e}$ and therefore also $\overset{\circ}{\mathbf{H}}^{e}$ and $\overset{+}{\mathbf{H}}^{e}$ differ only by pure rotation, as is seen from eqn (41).

Now introducing $\overset{+}{\mathbf{H}}^{e}$ according to (71) into (77) and determining the back-rotated Kirchhoff stress tensor $\overset{20}{\boldsymbol{\tau}}$ due to eqn (76) we obtain

$$\overset{20}{\boldsymbol{\tau}} = \frac{\partial W(\overset{+}{\mathbf{H}}^{e}, T)}{\partial \overset{+}{\mathbf{H}}^{e}} \cdot \frac{\partial \overset{+}{\mathbf{H}}^{e}}{\partial \overset{2*}{\mathbf{H}}^{e}} = \overset{+}{\boldsymbol{\tau}} \cdot \frac{\partial \overset{+}{\mathbf{H}}^{e}}{\partial \overset{2*}{\mathbf{H}}^{e}}, \tag{79}$$

with

$$\overset{+}{\boldsymbol{\tau}} := \frac{\partial W(\overset{+}{\mathbf{H}}^{e}, T)}{\partial \overset{+}{\mathbf{H}}}. \tag{80}$$

The partial derivative in (79) can be computed with respect to $\overset{2*}{\mathbf{H}}^{e}$ instead of $\overset{20}{\mathbf{H}}$, because both refer to the same reference configuration $\overset{10}{\mathfrak{B}}{}^{cp}$, and $\overset{2*}{\mathbf{H}}^{e}$ and $\overset{2*}{\mathbf{H}}^{p}$ are additive (57). The expression $\partial \overset{+}{\mathbf{H}}^{e} / \partial \overset{2*}{\mathbf{H}}^{e}$ can be evaluated numerically according to (72) and (73).

The rate form of eqn (79) can be written as

$$\begin{aligned}
\dot{\overset{20}{\boldsymbol{\tau}}} &= \left[\left(\frac{\partial \overset{+}{\mathbf{H}}^{e}}{\partial \overset{2*}{\mathbf{H}}^{e}} \right)^{\mathrm{T}} \cdot \frac{\partial^2 W(\overset{+}{\mathbf{H}}^{e}, T)}{\partial \overset{+}{\mathbf{H}}^{e} \otimes \partial \overset{+}{\mathbf{H}}^{e}} \cdot \frac{\partial \overset{+}{\mathbf{H}}^{e}}{\partial \overset{2*}{\mathbf{H}}^{e}} + \frac{\partial W(\overset{+}{\mathbf{H}}^{e}, T)}{\partial \overset{+}{\mathbf{H}}^{e}} \cdot \frac{\partial^2 \overset{+}{\mathbf{H}}^{e}}{\partial \overset{2*}{\mathbf{H}}^{e} \otimes \partial \overset{2*}{\mathbf{H}}^{e}} \right] \cdot \dot{\overset{2*}{\mathbf{H}}}{}^{e} \\
&= \left[\left(\frac{\partial \overset{+}{\mathbf{H}}^{e}}{\partial \overset{2*}{\mathbf{H}}^{e}} \right)^{\mathrm{T}} \cdot \mathbb{C}^{e} \cdot \frac{\partial \overset{+}{\mathbf{H}}^{e}}{\partial \overset{2*}{\mathbf{H}}^{e}} + \overset{+}{\boldsymbol{\tau}} \cdot \frac{\partial^2 \overset{+}{\mathbf{H}}^{e}}{\partial \overset{2*}{\mathbf{H}}^{e} \otimes \partial \overset{2*}{\mathbf{H}}^{e}} \right] \cdot \dot{\overset{2*}{\mathbf{H}}}{}^{e} \\
&= \overset{20}{\mathbb{C}}{}^{e} \cdot \dot{\overset{2*}{\mathbf{H}}}{}^{e},
\end{aligned} \tag{81}$$

where in (81)

$$\mathbb{C}^{e} := \frac{\partial^2 W(\overset{+}{\mathbf{H}}^{e}, T)}{\partial \overset{+}{\mathbf{H}}^{e} \otimes \partial \overset{+}{\mathbf{H}}^{e}} = \mathbb{C}^{e}(\overset{+}{\mathbf{H}}^{e}, T) \tag{82}$$

is the tangential elasticity tensor corresponding to $\overset{+}{\mathbf{H}}^{e}$ and

$$\overset{20}{\mathbb{C}}{}^{e} := \left[\left(\frac{\partial \overset{+}{\mathbf{H}}^{e}}{\partial \overset{2*}{\mathbf{H}}^{e}} \right)^{\mathrm{T}} \cdot \mathbb{C}^{e} \cdot \frac{\partial \overset{+}{\mathbf{H}}^{e}}{\partial \overset{2*}{\mathbf{H}}^{e}} + \overset{+}{\boldsymbol{\tau}} \cdot \frac{\partial^2 \overset{+}{\mathbf{H}}^{e}}{\partial \overset{2*}{\mathbf{H}}^{e} \otimes \partial \overset{2*}{\mathbf{H}}^{e}} \right] \tag{83}$$

is the "actual tangential elasticity tensor" associated with $\overset{2*}{\mathbf{H}}^{e}$ and $\overset{20}{\mathbf{H}}$.

Essential simplifications are possible, if the elastic strains are limited. According to the investigation of Anand (1979, 1986) the elastic strain energy for many materials (including metals and rubber-like materials) can be expressed for elastic stretches between 0.7 and 1.3 by a quadratic form, provided the strain tensor is replaced by the logarithmic (Hencky) strain tensor. Thus we can write

$$W(\overset{+}{\mathbf{H}}^{e}) = \tfrac{1}{2} \overset{+}{\mathbf{H}}^{e} \cdot \mathbb{C}^{e} \cdot \overset{+}{\mathbf{H}}^{e}, \tag{84}$$

with

$$\mathbb{C}^e = \text{const. for } |\overset{+}{H}{}^e_i| \leqslant 0.3, \tag{85}$$

where $\overset{+}{H}{}^e_i$ are the eigenvalues of $\overset{+}{\mathbf{H}}{}^e$.

If the elastic strains are of moderate order $0(\theta) < \frac{1}{10}$ such that the first deformation contains only plastic stretch, $\partial\overset{+}{\mathbf{H}}{}^e/\partial\overset{*}{\mathbf{H}}{}^e = \mathbf{11}$ (fourth order unity) and $\partial^2\overset{+}{\mathbf{H}}{}^e/\partial\overset{*}{\mathbf{H}}{}^e \otimes \partial\overset{*}{\mathbf{H}}{}^e = \mathbf{0}$. In this case

$$\overset{20}{\mathbb{C}}{}^e = \mathbb{C}^e \tag{86}$$

and

$$\overset{20}{\boldsymbol{\tau}} = \overset{+}{\boldsymbol{\tau}} = \mathbb{C}^e \cdot \overset{+}{\mathbf{H}}{}^e. \tag{87}$$

These equations remain approximately valid provided the largest difference of the eigenvalues of $\overset{+}{\mathbf{H}}{}^e$ (or $\mathring{\mathbf{H}}^e = \ln \mathring{\mathbf{U}}^e$) does not exceed 0.20.

The same statement is valid for

$$\dot{\overset{20}{\boldsymbol{\tau}}} = \mathbb{C}^e \cdot \dot{\overset{2*}{\mathbf{H}}}{}^e. \tag{88}$$

When the largest eigenvalue of $\overset{+}{\mathbf{H}}{}^e$ (or $\mathring{\mathbf{H}}^e$) also remains below 0.20, the error introduced in (88) remains below 5%.

4.2. *Elastic–plastic constitutive equations*

In order to formulate a yield condition and plastic flow rule suitable for the thermodynamical requirements we take into account the results of Le and Stumpf (1993). According to this paper the free energy ψ per unit mass is a function of the total elastic strain, it means a function of $\mathring{\mathbf{H}}^e$, of the temperature T and the metric tensor $\mathring{\mathbf{g}}^p$ of the configuration $\mathring{\mathfrak{B}}^p$, to which $\mathring{\mathbf{U}}^e = \exp \mathring{\mathbf{H}}^e$ is applied:

$$\psi = \psi(\mathring{\mathbf{H}}^e, T, (\mathring{\mathbf{g}}^p)^{-1}). \tag{89}$$

If we want to refer to the configuration $\overset{10}{\mathfrak{B}}{}^{ep}$ we have to consider the free energy function

$$\overset{20}{\psi} = \overset{20}{\psi}(\overset{2*}{\mathbf{H}}{}^e, \overset{10}{\mathbf{H}}{}^e, T, (\overset{10}{\mathbf{g}}{}^{ep})^{-1}), \tag{90}$$

where $\overset{10}{\mathbf{H}}{}^e = \ln \overset{10}{\mathbf{U}}{}^e$ is the logarithmic elastic strain tensor given by the first deformation, and $\overset{10}{\mathbf{g}}{}^{ep}$ is the metric tensor of the configuration $\overset{10}{\mathfrak{B}}{}^{ep}$, also given by the first deformation. If the elastic part of the material behavior is isotropic and remains so during the deformation the elastic strain state is sufficiently described by $\overset{2*}{\mathbf{H}}{}^e$ and $\overset{10}{\mathbf{H}}{}^e$. They uniquely determine $\overset{+}{\mathbf{H}}{}^e$ (71), which can be used to express the free energy of isotropically elastic materials.

Since the first deformation is fixed and known by preceding calculations $\overset{10}{\mathbf{g}}{}^{ep}$ and $\overset{10}{\mathbf{H}}{}^e$ have constant values, and the material time derivative $D_t(\cdot)$ is equal to the partial time derivative $(\dot{\ })$. Thus we obtain

$$D_t \overset{20}{\psi} = \dot{\overset{20}{\psi}} = \frac{\partial \overset{20}{\psi}}{\partial \overset{2*}{\mathbf{H}}{}^e} \cdot \dot{\overset{2*}{\mathbf{H}}}{}^e + \frac{\partial \overset{20}{\psi}}{\partial T} \dot{T}. \tag{91}$$

Next the Clausius–Duhem inequality according to eqn (3.28) in Le and Stumpf (1993) is considered. Choosing $\overset{10}{\mathfrak{B}}{}^{ep}$ as the reference configuration for the entropy production inequality we have

$$\rho(\eta\dot{T}+\overset{2\dot{0}}{\psi})-\overset{20}{\boldsymbol{\tau}}\cdot\overset{2\dot{0}}{\mathbf{H}}+\frac{1}{T}\operatorname{div}(T)\cdot\mathbf{q}\leqslant 0, \tag{92}$$

where ρ is the mass density, η the entropy, $\mathbf{q}$ the heat-flow and div the divergence operator, all with respect to the configuration $\overset{10}{\mathfrak{B}}{}^{\mathrm{ep}}$. Introducing (57) and (91) into (92) and varying $\overset{2\dot{*}}{\mathbf{H}}$, $\overset{2\dot{*}}{\mathbf{H}}$, T and $\dot{T}$ independently the constitutive equations

$$\overset{20}{\boldsymbol{\tau}}=\rho\frac{\partial\overset{20}{\psi}}{\partial\overset{2*}{\mathbf{H}}{}^{\mathrm{e}}}=\frac{\partial W}{\partial\overset{2*}{\mathbf{H}}{}^{\mathrm{e}}} \tag{93}$$

and the plastic dissipation

$$D^{\mathrm{p}}=\overset{20}{\boldsymbol{\tau}}\cdot\overset{2\dot{*}}{\mathbf{H}}{}^{\mathrm{p}}\geqslant 0 \tag{94}$$

can be derived, where $W=\rho\overset{20}{\psi}$ is the free energy per undeformed unit volume.

We assume that the yield function F depends on the actual stress state, the temperature field and a set of internal parameters as the average isotropic yield stress τ_y, and the so-called backstress tensor $\boldsymbol{\alpha}$ [see also Duszek and Perzyna (1991)]. Thus neglecting for simplicity thermal and damage effects the yield function F can be expressed as

$$F=F(\overset{20}{\boldsymbol{\tau}},\tau_y,\overset{20}{\boldsymbol{\alpha}}), \tag{95}$$

where the back-rotated Kirchhoff stress tensor $\overset{20}{\boldsymbol{\tau}}$ and the backstress tensor $\overset{20}{\boldsymbol{\alpha}}$ are directionally referred to the configuration $\overset{10}{\mathfrak{B}}{}^{\mathrm{ep}}$.

With (94) and (95) the associated flow rule can be derived using the maximum principle of plastic dissipation (94) yielding

$$\overset{2\dot{*}}{\mathbf{H}}{}^{\mathrm{p}}=\lambda\frac{\partial F}{\partial\overset{20}{\boldsymbol{\tau}}},\quad \lambda F=0,\quad \lambda\geqslant 0\quad\text{and}\quad F\leqslant 0, \tag{96}$$

with $F<0$ denoting no yielding and $F=0$ denoting yielding.

Introducing the von Mises yield criterion the yield function F can be given in the form

$$\mathbf{F}=\sqrt{\tilde{\boldsymbol{\tau}}\cdot\tilde{\boldsymbol{\tau}}}-\sqrt{\tfrac{2}{3}}\tau_y\begin{cases}<0\Leftrightarrow\text{no yielding}\\=0\Leftrightarrow\text{yielding}\end{cases} \tag{97}$$

with

$$\tilde{\boldsymbol{\tau}}=\overset{20}{\boldsymbol{\tau}}-\tfrac{1}{3}(\overset{20}{\boldsymbol{\tau}}\cdot\mathbf{1})\mathbf{1}-\overset{20}{\boldsymbol{\alpha}}, \tag{98}$$

where τ_y is the one-dimensional yield stress. The back-stress tensor $\overset{20}{\boldsymbol{\alpha}}$ must be referred to the same configuration as $\overset{20}{\boldsymbol{\tau}}$. This necessitates a rotational transformation of $\overset{20}{\boldsymbol{\alpha}}$ according to (111), when the first deformation is updated.

From (97) and (98) it follows that

$$\frac{\partial F}{\partial\overset{20}{\boldsymbol{\tau}}}=\frac{\tilde{\boldsymbol{\tau}}}{\sqrt{\tilde{\boldsymbol{\tau}}\cdot\tilde{\boldsymbol{\tau}}}}=:F_{,\overset{20}{\tau}}, \tag{99}$$

$$\frac{\partial F}{\partial \overset{20}{\boldsymbol{\alpha}}} = -\frac{\tilde{\boldsymbol{\tau}}}{\sqrt{\tilde{\boldsymbol{\tau}}\cdot\tilde{\boldsymbol{\tau}}}} =: F_{,\overset{20}{\alpha}} \tag{100}$$

and

$$\frac{\partial F}{\partial \tau_{\mathrm{y}}} = -\sqrt{\tfrac{2}{3}} =: F_{,\tau_{\mathrm{y}}}. \tag{101}$$

Taking into account results of Duszek and Perzyna (1991) the following evolution laws for $\overset{20}{\boldsymbol{\alpha}}$ and τ_{y} are suggested, where we use the associated flow rule (96)

$$\overset{\dot{20}}{\boldsymbol{\alpha}} = \zeta \overset{\dot{2*}}{\mathbf{H}}{}^{\mathrm{p}}, \quad \dot{\tau}_{\mathrm{y}} = \left(h_1 \frac{F_{,\overset{20}{\tau}}}{\| F_{,\overset{20}{\tau}} \|} + h_2 \overset{20}{\boldsymbol{\alpha}} \right) \cdot \overset{\dot{2*}}{\mathbf{H}}{}^{\mathrm{p}}. \tag{102}$$

Here ζ, h_1 and h_2 are material properties which may depend on the total plastic stretch $\mathbf{U}^{\mathrm{p}}$. A refined evolution law for $\overset{20}{\boldsymbol{\alpha}}$ using the exactly computed plastic spin with reference to the configuration $\overset{10}{\mathfrak{B}}{}^{\mathrm{ep}}$ will be considered in a forthcoming paper. Generalizations within the framework of extended thermodynamics are proposed in Stumpf and Badur (1992).

Inserting (96) into (81) and using (64)

$$\overset{\dot{20}}{\boldsymbol{\tau}} = \overset{20}{\mathbb{C}}{}^{\mathrm{e}} \cdot (\overset{\dot{20}}{\mathbf{H}} - \lambda F_{,\overset{20}{\tau}}) \tag{103}$$

is derived, where thermal strains are not considered for simplicity. During the process of yielding $F = 0$ has to hold, thus the condition

$$\dot{F} = F_{,\overset{20}{\tau}} \cdot \overset{\dot{20}}{\boldsymbol{\tau}} + F_{,\overset{2*}{\mathbf{H}}{}^{\mathrm{p}}} \cdot \overset{\dot{2*}}{\mathbf{H}}{}^{\mathrm{p}} = 0 \tag{104}$$

has to be satisfied, where

$$F_{,\overset{2*}{\mathbf{H}}{}^{\mathrm{p}}} := \frac{\partial F}{\partial \overset{2*}{\mathbf{H}}{}^{\mathrm{p}}} = F_{,\overset{20}{\alpha}} \cdot \frac{\partial \overset{\dot{20}}{\boldsymbol{\alpha}}}{\partial \overset{\dot{2*}}{\mathbf{H}}{}^{\mathrm{p}}} + F_{,\tau_{\mathrm{y}}} \frac{\partial \dot{\tau}_{\mathrm{y}}}{\partial \overset{\dot{2*}}{\mathbf{H}}{}^{\mathrm{p}}}. \tag{105}$$

Equations (96) and (103) are inserted into (104) yielding

$$F_{,\overset{20}{\tau}} \cdot \overset{20}{\mathbb{C}}{}^{\mathrm{e}} \cdot (\overset{\dot{20}}{\mathbf{H}} - \lambda F_{,\overset{20}{\tau}}) + F_{,\overset{2*}{\mathbf{H}}{}^{\mathrm{p}}} \cdot \lambda F_{,\overset{20}{\tau}} = 0. \tag{106}$$

Solving this equation with respect to λ

$$\lambda = \frac{F_{,\overset{20}{\tau}} \cdot \overset{20}{\mathbb{C}}{}^{\mathrm{e}} \cdot \overset{\dot{20}}{\mathbf{H}}}{F_{,\overset{20}{\tau}} \cdot \overset{20}{\mathbb{C}}{}^{\mathrm{e}} \cdot F_{,\overset{20}{\tau}} - F_{,\overset{2*}{\mathbf{H}}{}^{\mathrm{p}}} \cdot F_{,\overset{20}{\tau}}} \tag{107}$$

is obtained. Introducing (107) into (103) results in

$$\overset{\dot{20}}{\boldsymbol{\tau}} = \overset{20}{\mathbb{C}}{}^{\mathrm{ep}} \cdot \overset{\dot{20}}{\mathbf{H}}, \tag{108}$$

where

$$\overset{20}{\mathbb{C}}{}^{ep} = \overset{20}{\mathbb{C}}{}^{e} - \frac{(\overset{20}{\mathbb{C}}{}^{e} \cdot F_{,\overset{20}{\tau}}) \otimes (F_{,\overset{20}{\tau}} \cdot \overset{20}{\mathbb{C}}{}^{e})}{F_{,\overset{20}{\tau}} \cdot \overset{20}{\mathbb{C}}{}^{e} \cdot F_{,\overset{20}{\tau}} - F_{,\overset{2*}{\mathbf{H}^p}} \cdot F_{,\overset{20}{\tau}}} \tag{109}$$

is the elastic–plastic material tensor. In an incremental formulation we obtain from (109)

$$\Delta\overset{20}{\boldsymbol{\tau}} = \overset{20}{\mathbb{C}}{}^{ep} \cdot \Delta\overset{20}{\mathbf{H}}, \tag{110}$$

where $\Delta\overset{20}{\boldsymbol{\tau}} = \dot{\overset{20}{\boldsymbol{\tau}}}\,\Delta t$ is the stress increment, $\Delta\overset{20}{\mathbf{H}} = \dot{\overset{20}{\mathbf{H}}}\,\Delta t$ the strain increment and Δt a small positive time increment. Equations (109), (110) are used to establish the consistent tangential stiffness matrix in a FEM program.

Equations of the type (95)–(110) are already well known from the non-linear small strain analysis. There we have to replace the Green strain tensor by the logarithmic strain tensor of the second deformation and the second Piola–Kirchhoff stress tensor by the back-rotated Kirchhoff stress tensor. Therefore subroutines in common use, which model the elastic–plastic material behavior, can be employed.

When the first deformation is updated, the reference configuration $\overset{10}{\mathfrak{B}}{}^{ep}$ for $\overset{20}{\boldsymbol{\tau}}$ and $\overset{20}{\boldsymbol{\alpha}}$, is changed (see Section 2.3). Thus they have to be rotated using the rotation tensor $\overset{\#}{\mathbf{R}}$ according to (44), which describes the change of the directional orientation of the configuration $\overset{10}{\mathfrak{B}}{}^{ep}$ during the updating, resulting in

$$\overset{20}{\boldsymbol{\tau}}_a = \overset{\#}{\mathbf{R}}\,\overset{20}{\boldsymbol{\tau}}_b\,\overset{\#}{\mathbf{R}}{}^T, \quad \overset{20}{\boldsymbol{\alpha}}_a = \overset{\#}{\mathbf{R}}\,\overset{20}{\boldsymbol{\alpha}}_b\,\overset{\#}{\mathbf{R}}{}^T, \tag{111}$$

where $(\cdot)_b$ and $(\cdot)_a$ indicate tensors before and after the update of their reference configuration

If the stresses $\overset{20}{\boldsymbol{\tau}}$ are computed directly from the elastic stains using eqn (79) the transformation $(111)_1$ is superfluous because it is implicitly fulfilled as a result of the directional changes of $\overset{+}{\mathbf{H}}{}^e$ and $\overset{2*}{\mathbf{H}}{}^e$. But the back-stress tensor $\overset{20}{\boldsymbol{\alpha}}$ has always to be transformed explicitly.

5. PRINCIPLE OF VIRTUAL WORK, BOUNDARY CONDITIONS AND ASPECTS OF NUMERICAL REALIZATION

The principle of virtual work as a weak form of the equilibrium equations and static boundary conditions can be used to derive an efficient finite element algorithm. As basic variables of a virtual work functional suitable for the large strain analysis of this paper we use the back-rotated Kirchhoff stress tensor $\overset{20}{\boldsymbol{\tau}}$ and the variation of the logarithmic strain tensor of the second deformation $\delta\overset{20}{\mathbf{H}}$. Within our approximation they are work-conjugate and referred to the initial (undeformed) volume $\mathring{V}$. Thus the internal virtual work IVW results in

$$IVW = \int_{\mathring{V}} \overset{20}{\boldsymbol{\tau}} \cdot \delta\overset{20}{\mathbf{H}}\,\mathrm{d}\mathring{V}. \tag{112}$$

The external virtual work EVW is given by the volume forces $\mathbf{p}$ and surface traction $\mathbf{P}$, which are work-conjugate to the variation of the displacement field $\delta\mathbf{u}$, both referred to the actual volume V and actual surface S, respectively, resulting in

$$EVW = -\int_V \mathbf{p}\cdot\delta\mathbf{u}\,\mathrm{d}V - \int_S \mathbf{P}\cdot\delta\mathbf{u}\,\mathrm{d}S. \tag{113}$$

With the Jacobian, J, the conservation of mass yields

$$J = \frac{\rho_0}{\rho} = \frac{\mathrm{d}V}{\mathrm{d}\mathring{V}}, \tag{114}$$

where ρ_0 and ρ are the mass densities of the initial and actual configuration, respectively. We introduce volume forces of Kirchhoff-type $\mathring{\mathbf{p}}$ referred to the initial volume $\mathring{V}$

$$\mathring{\mathbf{p}} := J\mathbf{p} = \mathbf{p}\frac{\mathrm{d}V}{\mathrm{d}\mathring{V}} \tag{115}$$

and analogous surface tractions related to the initial surface $\mathring{S}$ defined by

$$\mathring{\mathbf{P}} := \mathbf{P}\frac{\mathrm{d}S}{\mathrm{d}\mathring{S}}. \tag{116}$$

Introducing (115) and (116) into (113) yields

$$EVW = -\int_{\mathring{V}} \mathring{\mathbf{p}}\cdot\delta\mathbf{u}\,\mathrm{d}\mathring{V} - \int_{\mathring{S}} \mathring{\mathbf{P}}\cdot\delta\mathbf{u}\,\mathrm{d}\mathring{S}. \tag{117}$$

With the internal virtual work (112) and the external virtual work (117) the principle of virtual work can be formulated as follows:

$$VW = IVW + EVW = \int_{\mathring{V}} (\overset{20}{\boldsymbol{\tau}}\cdot\delta\overset{20}{\mathbf{H}} - \mathring{\mathbf{p}}\cdot\delta\mathbf{u})\,\mathrm{d}\mathring{V} - \int_{\mathring{S}} \mathring{\mathbf{P}}\cdot\delta\mathbf{u}\,\mathrm{d}\mathring{S} = 0 \tag{118}$$

for all geometrically admissible variations $\delta\mathbf{u}$ satisfying homogeneous geometric boundary conditions on that part of the boundary, where geometrical quantities are prescribed.

To derive the Euler-Lagrange equations of (118) we introduce (68), (70) and (75) into (118) and obtain

$$\begin{aligned} VW &= \int_{\mathring{V}} [(\overset{20}{\mathbf{R}^{\mathrm{T}}}\boldsymbol{\tau}\overset{20}{\mathbf{R}})\cdot(\overset{20}{\mathbf{R}^{\mathrm{T}}}\delta_{\mathfrak{L}}\mathbf{e}\overset{20}{\mathbf{R}}) - \mathring{\mathbf{p}}\cdot\delta\mathbf{u}]\,\mathrm{d}\mathring{V} - \int_{\mathring{S}} \mathring{\mathbf{P}}\cdot\delta\mathbf{u}\,\mathrm{d}\mathring{S} \\ &= \int_{\mathring{V}} [\boldsymbol{\tau}\cdot\delta_{\mathfrak{L}}\mathbf{e} - \mathring{\mathbf{p}}\cdot\delta\mathbf{u}]\,\mathrm{d}\mathring{V} - \int_{\mathring{S}} \mathring{\mathbf{P}}\cdot\delta\mathbf{u}\,\mathrm{d}\mathring{S}. \end{aligned} \tag{119}$$

Expressing in (119) the Kirchhoff stress tensor $\boldsymbol{\tau}$ by the Cauchy stress tensor $\boldsymbol{\sigma}$

$$\boldsymbol{\tau} = J\boldsymbol{\sigma} \tag{120}$$

and using (115) and (116) eqn (119) can be obtained as

$$VW = \int_V (\boldsymbol{\sigma}\cdot\delta_{\mathfrak{L}}\mathbf{e} - \mathbf{p}\cdot\delta\mathbf{u})\,\mathrm{d}V - \int_S (\mathbf{P}\cdot\delta\mathbf{u})\,\mathrm{d}S = 0. \tag{121}$$

In (121) we transform the Lie-variation of the Almansi strain tensor as follows:

$$\begin{aligned} \delta_{\mathfrak{L}}\mathbf{e} &= \mathbf{F}^{-\mathrm{T}}\delta(\mathbf{F}^{\mathrm{T}}\mathbf{e}\mathbf{F})\mathbf{F}^{-1} = \tfrac{1}{2}(\mathbf{F}^{-\mathrm{T}}\delta\mathbf{F}^{\mathrm{T}} + \delta\mathbf{F}\,\mathbf{F}^{-1}) \\ &= \frac{1}{2}\left[\left(\frac{\partial\mathbf{X}}{\partial\mathbf{x}}\right)^{\mathrm{T}}\left(\frac{\partial(\delta\mathbf{u})}{\partial\mathbf{X}}\right)^{\mathrm{T}} + \frac{\partial(\delta\mathbf{u})}{\partial\mathbf{X}}\frac{\partial\mathbf{X}}{\partial\mathbf{x}}\right] \\ &= \frac{1}{2}\left[\left(\frac{\partial(\delta\mathbf{u})}{\partial\mathbf{x}}\right)^{\mathrm{T}} + \frac{\partial(\delta\mathbf{u})}{\partial\mathbf{x}}\right] = \tfrac{1}{2}[(\delta\mathbf{u})\nabla + \nabla(\delta\mathbf{u})], \end{aligned} \tag{122}$$

where $\mathbf{x}$ and $\mathbf{X}$ are position vectors of material points in the actual configuration $\mathfrak{B} = \overset{t}{\mathfrak{B}}$ and in the initial configuration $\overset{\circ}{\mathfrak{B}}$. With $\mathbf{u}$ we denote the total displacement field and with ∇ the gradient operator with respect to the actual configuration.

Introducing (122) into (121) and applying standard variational technique we obtain the local equilibrium equation

$$\operatorname{div} \boldsymbol{\sigma} + \mathbf{p} = 0 \quad \text{in } V \tag{123}$$

and the static boundary conditions

$$\boldsymbol{\sigma}\mathbf{n} - \mathbf{P} = 0 \quad \text{on } S_p \tag{124}$$

as Euler–Lagrange equations, where $\mathbf{n}$ denotes the outer unit normal vector on S, and S_p denotes that part of S, where static quantities are prescribed. The complementary geometrical boundary condition is

$$\mathbf{u} = \mathbf{u}^* \quad \text{on } S_u, \tag{125}$$

where $\mathbf{u}^*$ is the prescribed displacement on S_u.

The incremental form of (118) allowing the construction of the consistent tangential operator for an appropriate finite element algorithm can be derived as Gâteaux differential of the virtual work functional VW according to (118):

$$\begin{aligned} \Delta VW &= \Delta VW(\mathbf{u}, \Delta\mathbf{u}; \delta\mathbf{u}) \\ &= \int_{\mathring{V}} (\overset{20}{\boldsymbol{\tau}} \cdot \Delta\delta\overset{20}{\mathbf{H}} + \Delta\overset{20}{\boldsymbol{\tau}} \cdot \delta\overset{20}{\mathbf{H}} - \Delta\mathring{\mathbf{p}} \cdot \delta\mathbf{u})\, \mathrm{d}\mathring{V} - \int_{\mathring{S}} \Delta\mathring{\mathbf{P}} \cdot \delta\mathbf{u}\, \mathrm{d}\mathring{S}, \end{aligned} \tag{126}$$

where Δ and δ indicate independent variations of the displacement field $\mathbf{u}$. The variations $\delta\overset{20}{\mathbf{H}}$, $\Delta\delta\overset{20}{\mathbf{H}}$ and $\Delta\overset{20}{\boldsymbol{\tau}}$ are given by (62), (63) and (81) or (110), according to elastic or plastic material response.

The virtual work functional (118) can be expanded in a Taylor series yielding

$$VW(\mathbf{u}+\Delta\mathbf{u}; \delta\mathbf{u}) = VW(\mathbf{u}; \delta\mathbf{u}) + \Delta VW(\mathbf{u}, \Delta\mathbf{u}; \delta\mathbf{u}) + \cdots = 0, \tag{127}$$

where the first term is given by the functional (118) and the second term by (126).

Let us introduce nodal displacements q_i, $i \in \{1, 2, 3, \ldots, n\}$ (nodal degrees of freedom) determining the interpolation of the displacement field

$$\mathbf{u} = \mathbf{u}(q_i), \quad i = 1, 2, \ldots, n. \tag{128}$$

Then we obtain the variational quantities

$$\begin{aligned} \delta\mathbf{u} &= \frac{\partial \mathbf{u}}{\partial q_i}\, \delta q_i, \quad i = 1, \ldots, n, \\ \Delta\mathbf{u} &= \frac{\partial \mathbf{u}}{\partial q_j}\, \Delta q_j, \quad j = 1, \ldots, n, \\ \delta\overset{20}{\mathbf{H}} &= \delta\overset{20}{\mathbf{H}}(\mathbf{u}; \delta\mathbf{u}(\delta q_i)) := \frac{\partial \overset{20}{\mathbf{H}}}{\partial \mathbf{u}}\, \delta\mathbf{u}, \\ \Delta\delta\overset{20}{\mathbf{H}} &= \Delta\delta\overset{20}{\mathbf{H}}(\mathbf{u}; \Delta\mathbf{u}(\Delta q_j), \quad \delta\mathbf{u}(\delta q_i)) := \frac{\partial^2 \overset{20}{\mathbf{H}}}{\partial \mathbf{u} \otimes \partial \mathbf{u}}\, \Delta\mathbf{u}\, \delta\mathbf{u}, \end{aligned} \tag{129}$$

$$\Delta\overset{20}{\boldsymbol{\tau}} = \begin{cases} \overset{20}{\mathbb{C}^{e}}\cdot \Delta\overset{20}{\mathbf{H}}(\mathbf{u};\Delta\mathbf{u}(\Delta q_j)) & \text{for elastic response,} \\ \overset{20}{\mathbb{C}^{ep}}\cdot \Delta\overset{20}{\mathbf{H}}(\mathbf{u};\Delta\mathbf{u}(\Delta q_j)) & \text{for plastic response,} \end{cases}$$

$$= \frac{\partial \overset{20}{\boldsymbol{\tau}}}{\partial \overset{20}{\mathbf{H}}}\cdot\frac{\partial \overset{20}{\mathbf{H}}}{\partial \mathbf{u}}\Delta\mathbf{u} \quad \text{in general.} \tag{130}$$

With (129), (130) eqn (127) takes the form

$$VW = RFV_i\,\delta q_i + STM_{ij}\,\Delta q_j\,\delta q_i = 0 \tag{131}$$

or equivalently

$$STM_{ij}\,\Delta q_j + RFV_i = 0, \tag{132}$$

where higher order terms were neglected. In (131), (132) i and j are running from 1 to n with summation over indices appearing twice in a term. In (131) and (132), respectively,

$$RFV_i = \int_{\mathring{V}}\left(\overset{20}{\boldsymbol{\tau}}\cdot\frac{\partial \overset{20}{\mathbf{H}}}{\partial \mathbf{u}}\frac{\partial \mathbf{u}}{\partial q_i} - \mathring{\mathbf{p}}\cdot\frac{\partial \mathbf{u}}{\partial q_i}\right)\mathrm{d}\mathring{V} - \int_{\mathring{S}}\mathring{\mathbf{P}}\cdot\frac{\partial \mathbf{u}}{\partial q_i}\,\mathrm{d}\mathring{S} \tag{133}$$

is the residual force vector and

$$STM_{ij} = \int_{\mathring{V}}\left\{\overset{20}{\boldsymbol{\tau}}\cdot\left[\left(\frac{\partial^2 \overset{20}{\mathbf{H}}}{\partial \mathbf{u}\otimes\partial \mathbf{u}}\frac{\partial \mathbf{u}}{\partial q_j}\right)\cdot\frac{\partial \mathbf{u}}{\partial q_i}\right] + \left(\frac{\partial \overset{20}{\boldsymbol{\tau}}}{\partial \overset{20}{\mathbf{H}}}\cdot\frac{\partial \overset{20}{\mathbf{H}}}{\partial \mathbf{u}}\frac{\partial \mathbf{u}}{\partial q_j}\right)\cdot\left(\frac{\partial \overset{20}{\mathbf{H}}}{\partial \mathbf{u}}\frac{\partial \mathbf{u}}{\partial q_i}\right) - \left(\frac{\partial \mathring{\mathbf{p}}}{\partial \mathbf{u}}\frac{\partial \mathbf{u}}{\partial q_j}\right)\cdot\frac{\partial \mathbf{u}}{\partial q_i}\right\}\mathrm{d}\mathring{V} - \int_{\mathring{S}}\left(\frac{\partial \mathring{\mathbf{P}}}{\partial \mathbf{u}}\frac{\partial \mathbf{u}}{\partial q_j}\right)\cdot\frac{\partial \mathbf{u}}{\partial q_i}\,\mathrm{d}\mathring{S} \tag{134}$$

the consistent tangent stiffness matrix.

Equations (132), (133) and (134) are basic formulae for the Newton–Raphson iteration scheme. In (132) Δq_j denote the corrections of the nodal displacements q_j, and RFV_i is going to be iterated to zero.

6. CONCLUSION

The introduction of the Lagrangean logarithmic strain tensor for a superposed moderately large strain deformation and its additive decomposition into purely elastic and purely plastic parts enables an appropriate formulation of isotropic-elastic and plastic constitutive and evolution equations with combined isotropic and kinematic hardening. All formulae are given to analyse deformations undergoing arbitrarily large elastic and arbitrarily large plastic strains. The presented concept can be used to formulate a solution algorithm, that can realize load steps corresponding to moderately large strains and unrestricted rotations. No assumptions are needed concerning the decomposition of the rotation into an elastic and a plastic rotation or concerning the magnitude of elastic and plastic spin. The presented model can be employed to derive a theory of shells undergoing large elastic and large plastic strains and finite rotations based on the shell theory of Schieck *et al.* (1992). This will be shown in a forthcoming paper (Stumpf and Schieck, 1993).

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